A simple counting argument

Matthieu Rosenfeld
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Local Lemmas and others
Lovász Local Lemma

Probabilistic method (Erdős):

Probabilistic argument $\implies$ deterministic result
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**Lovász Local Lemma (1975)**

Let $A_1, \ldots, A_k$ be events such that each event has probability at most $p$ and depends on at most $d$ other events. If

$$epd \leq 1$$

then the probability that no events occur is non-zero.
Probabilistic method (Erdős):
Probabilistic argument $\implies$ deterministic result

**Lovász Local Lemma (1975)**
Let $A_1, \ldots, A_k$ be events such that each event has probability at most $p$ and depends on at most $d$ other events. If

$$epd \leq 1$$

then the probability that no events occur is non-zero.

**Theorem**
*Let $\phi$ be a $k$-SAT formula. If every variable belongs to at most $\frac{2^k}{ke}$ clauses then $\phi$ is satisfiable.*
Theorem (Moser et Tardos, 2010, Gödel prize 2020)

Under LLL assumptions, there is a randomized algorithm that finds a satisfying assignment in expected polynomial time.

⇒ Entropy Compression
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⇒ Entropy Compression

⇒ improvement of many bounds previously obtained by the LLL (SAT, graph or hypergraph colorings, combinatorics on words...)

... Entropy compression, local-cut Lemma, Counting argument
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Other techniques: Local-cut Lemma, Cluster-expansion (from statistical physics)
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Under LLL assumptions, there is a randomized algorithm that finds a satisfying assignment in expected polynomial time.

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Other techniques: Local-cut Lemma, Cluster-expansion (from statistical physics)...

[Rosenfeld, 2020] Counting argument: this talk
A first example: proper hypergraph colorings
A coloring of the vertices of $H$ is **proper** if no edge is monochromatic.
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The **chromatic number** $\chi(H)$ is the minimum of colors in a proper coloring of $H$.

The **degree** of a vertex $v$ is the number of edges that contain $v$. 

An hypergraph is $r$-regular if every edge is of size $r$. 
A coloring of the vertices of $H$ is **proper** if no edge is monochromatic.

The **chromatic number** $\chi(H)$ is the minimum of colors in a proper coloring of $H$.

The **degree** of a vertex $v$ is the number of edges that contain $v$.

An hypergraph is **$r$-regular** if every edge is of size $r$. 
Hypergraph colorings: a result

Intuitively: smaller degree or larger edges $\implies$ easier to color

Theorem
Let $\beta \geq 1$. Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$,
$\chi(H) \leq \beta + \Delta$. Let $c=\beta+\Delta$ and $C(H)$ be a proper $c$-coloring of $H$.

Lemma
Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$, then $
\forall v \in V(H), C(H) \geq \beta C(H-v)$.
Hypergraph colorings: a result

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**Theorem**

Let \( \beta \geq 1 \). Let \( H \) be a \( r \)-uniform hypergraph of maximum degree \( \Delta \),

\[
\chi(H) \leq \left\lceil \beta + \frac{\Delta}{\beta^{r-2}} \right\rceil.
\]

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**Theorem**

Let $\beta \geq 1$. Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$, \[\chi(H) \leq \left\lceil \beta + \frac{\Delta}{\beta^{r-2}} \right\rceil.\]

Let $c = \left\lceil \beta + \frac{\Delta}{\beta^{r-2}} \right\rceil$ and $C(H)$ be $\#$ proper $c$-coloring of $H$. 
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Let $c = \left\lfloor \beta + \frac{\Delta}{\beta^{r-2}} \right\rfloor$ and $C(H)$ be the number of proper $c$-coloring of $H$.

**Lemma**

Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$, then

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\forall v \in V(H), \quad C(H) \geq \beta C(H - v).
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**Theorem**

Let \(\beta \geq 1\). Let \(H\) be a \(r\)-uniform hypergraph of maximum degree \(\Delta\),

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Let \(c = \left\lceil \beta + \frac{\Delta}{\beta^{r-2}} \right\rceil\) and \(C(H)\) be \# proper \(c\)-coloring of \(H\).

**Lemma**

Let \(H\) be a \(r\)-uniform hypergraph of maximum degree \(\Delta\), then

\[
\forall v \in V(H), \quad C(H) \geq \beta C(H - v).
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**Proof of the Theorem:** By induction, \(C(H) \geq \beta^{|H|}\)  

\(\square\)
Proof by induction that $\forall v \in V(H), \quad C(H) \geq \beta C(H - v)$. 

Induction hypothesis implies:

$\forall S \subseteq V(H - v), \quad C(H - v - S) \leq C(H - v) \beta |S|$. 

A coloring of $V(H)$ is bad, if it is proper on $H - v$, but not on $H$. 

$C(H) \geq c \cdot C(H - v) - \#\{\text{bad colorings}\} \geq c \cdot C(H - v) - \Delta C(H - v) \beta r - 2$. 

For each edge $e \sim v$, a bad $c$-coloring of $H$ is $e$-bad, if it is monochromatic on $e$. 

$\#\{\text{bad colorings}\} \leq X \#\{e\text{-bad colorings}\} \leq \Delta C(H - v) \beta r - 2$. 

Let $w \in e \setminus v$, then $\#\{e\text{-bad colorings}\} \leq C(H - (e \setminus w)) \leq C(H - v) \beta r - 2$. 

Finally, $C(H) \geq \frac{1}{2} c - \Delta \beta r - 2 \cdot C(H - v) \geq \beta C(H - v)$. 

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Let $w \in e \setminus v$, then

$\#\{e\text{-bad colorings}\} \leq C(H - (e \setminus w)) \leq \frac{C(H - v)}{\beta r - 2}$

Finally, $C(H) \geq \frac{C(H - v)}{\beta}$
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**Lemma**

Let $H$ be an $r$-regular hypergraph of maximum degree $\Delta$. Then

$$\chi(H) \leq c := \min_{\beta > 0} \left[ \beta + \frac{\Delta}{\beta^{r-2}} \right].$$

Moreover, the number of proper $c$-coloring of $H$ is at least $\beta |V(H)|$.

**Theorem (Wanless and Wood, 2020)**

Let $r > 2$. Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$, $\chi(H) \leq \frac{\Delta}{r-1} \frac{1}{(r-1)(r-2)}$.

Asymptotically optimal. Slightly better than [Erdős and Lovász, 1975]

**Remark:** For the chromatic number of graphs (2-regular hypergraph), we have $c = \Delta + 1$. 
**Lemma**

Let $H$ be an $r$-regular hypergraph of maximum degree $\Delta$. Then

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Moreover, the number of proper $c$-coloring of $H$ is at least $\beta^{|V(H)|}$. 
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Theorem (Wanless and Wood, 2020)

Let $r > 2$. Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$,

$$\chi(H) \leq \left[ \left( \frac{r-1}{r-2} \right) ((r - 2)\Delta)^{1/(r-1)} \right].$$

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Theorem (Wanless and Wood, 2020)

Let $r > 2$. Let $H$ be a $r$-uniform hypergraph of maximum degree $\Delta$,

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Remark: For the chromatic number of graphs (2-regular hypergraph), we have $c = \Delta + 1$. 
A second example: Star coloring
A star coloring of a graph $G$ is a proper coloring such that any pair of color classes induces a forest of stars.
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$\chi_s(G)$ is the minimum number of colors in a star coloring of $G$. 
For all graph $G$ of maximum degree $\Delta$, 

**Theorem (Fertin, Raspaud, and Reed, 2004)** 

$$\chi_s(G) \leq 20\Delta^{3/2}$$
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**Theorem (Ndreca, Procacci, and Scoppola, 2012)**

$$\chi_s(G) \leq 4.34\Delta^{3/2} + 1.5\Delta$$
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**Theorem (Esperet, and Parreau, 2013)**

$$\chi_s(G) \leq \left[ 2\sqrt{2}\Delta^{3/2} + \Delta \right]$$
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**Theorem (Esperet, and Parreau, 2013 Entropy Compression)**

$$\chi_s(G) \leq \left\lfloor 2\sqrt{2}\Delta^{3/2} + \Delta \right\rfloor$$
Lemma

For any graph $G$ of maximum degree $\Delta$ and any $v \in V(G)$, the number of $p_4$ that contains $v$ is at most,

$$|\{p | p \text{ is a } p_4, v \in p\}| \leq 2\Delta^3.$$
A useful lemma

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In total: $\leq \Delta^3$ choices
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Star coloring: with the counting argument

**Theorem (This talk)**

\[ \chi_s(G) \leq \left\lfloor 2\sqrt{2}\Delta^{3/2} + \Delta \right\rfloor \]
Theorem (This talk)

\[ \chi_s(G) \leq \left\lceil 2\sqrt{2}\Delta^{3/2} + \Delta \right\rceil \]

Let \( c = \left\lceil 2\sqrt{2}\Delta^{3/2} + \Delta \right\rceil \).

We let \( C_s(G) \) be the number of star \( c \)-colorings of \( G \).

Let \( \beta \) such that \( c - \Delta - \frac{2\Delta^3}{\beta} \geq \beta \).
Theorem (This talk)

\[ \chi_s(G) \leq \left\lfloor 2\sqrt{2\Delta^3/2} + \Delta \right\rfloor \]

Let \( c = \left\lfloor 2\sqrt{2\Delta^3/2} + \Delta \right\rfloor \).

We let \( C_s(G) \) be the number of star \( c \)-colorings of \( G \).

Let \( \beta \) such that \( c - \Delta - \frac{2\Delta^3}{\beta} \geq \beta \).

Theorem is a corollary of:

Lemma

For any graph \( G \) of maximum degree \( \Delta \) and any \( v \in V(G) \),

\[ C_s(G) \geq \beta C_s(G - v) \].
Proof by induction that $\forall v \in V(G), \quad C_s(G) \geq \beta C_s(G - v)$
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$|\{\text{bad col.}\}| \leq \sum_{p \in P_4, \, v \in p} |\{p\text{-bad col.}\}| \leq 2\Delta^3 \frac{C_s(G - v)}{\beta}$

Let $u \in N(v) \cap p$, then

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Finally, $C_s(G) \geq \left( c - \Delta - \frac{2\Delta^3}{\beta} \right) C_s(G - v) \geq \beta C_s(G - v) \quad \square$
Theorem

Let $G$ be a graph of maximum degree $\Delta$, then

$$\chi_s(G) \leq \min_{\beta > 0} \left\lfloor \Delta + \beta + \frac{2\Delta^3}{\beta} \right\rfloor$$
Theorem

Let $G$ be a graph of maximum degree $\Delta$, then

$$\chi_s(G) \leq \min_{\beta > 0} \left\lfloor \Delta + \beta + \frac{2\Delta^3}{\beta} \right\rfloor = \left\lfloor \Delta + 2\sqrt{2\Delta^{3/2}} \right\rfloor.$$
Other applications
Problem (Vizing, 68)

If $\Delta(G)$ is the maximum degree of a vertex in a graph $G$, it is clear that $\chi(G) \leq \Delta(G) + 1$. [...] Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.
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Similar problem already mentioned in the 50’s by different authors (Erdős, Mycielski, Zykov...).
## Theorem (Johanson, 1996)

*For any triangle-free graph $G$ of maximum degree $\Delta$,*

$$\chi(G) = O\left(\frac{\Delta}{\log \Delta}\right)$$
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Bernshteyn, Brazelton, Cao, and Kang, 2021

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• group theory,
• ...
Wanless and Wood framework

**Theorem (Wanless and Wood, 2020)**

Let \((G, B)\) be an instance. Assume there exist a real number \(\beta \geq 1\) and an integer \(c \geq 1\) such that for every vertex \(v\) of \(G\),

\[
c \geq \beta + \sum_{k \geq 0} \beta^{-k} E_k(v).
\]

Then \(G\) is \((B, c)\)-choosable. Moreover, for every \(c\)-list assignment \(L\) of \(G\),

\[
P(G, B, L) \geq \beta^{|V(G)|}.
\]
The counting argument

- Is easier to use than entropy compression (and provides lower bounds on the number of solutions)
- Frequently easier to use than LLL and frequently provide better bounds than LLL
- It seems that it is a particular case of the Local Cut Lemma, but it is much easier to use
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