A simple counting argument

Matthieu Rosenfeld

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Local Lemmas and others

Lovász Local Lemma

Probabilistic method (Erdős):

Probabilistic argument $\implies$ deterministic result

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Lovász Local Lemma (1975)

Let $A_1, \ldots, A_k$ be events such that each event has probability at most p and depends on at most d other events. If

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then the probability that no events occur is non-zero.

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Theorem

Let ϕ be a k-SAT formula. If every variable belongs to at most $\frac{2^k}{ke}$ clauses then ϕ is satisfiable.

Theorem (Moser et Tardos, 2010, Gödel prize 2020)

Under LLL assumptions, there is a randomized algorithm that finds a satisfying assignment in expected polynomial time.

$\implies$ Entropy Compression

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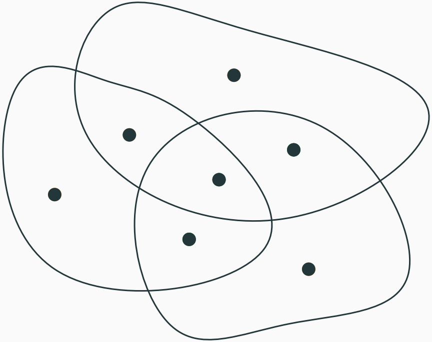
[**Rosenfeld**, 2020] Counting argument: this talk

A first example:
proper hypergraph colorings

Hypergraph colorings: Definition and Theorem

A coloring of the vertices of H is **proper** if no edge is monochromatic.



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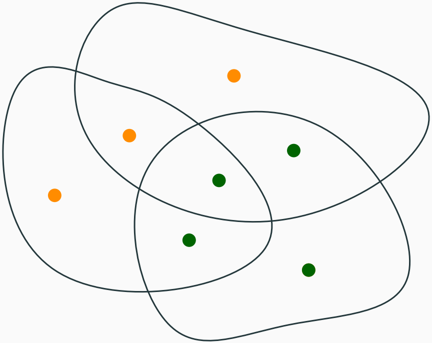

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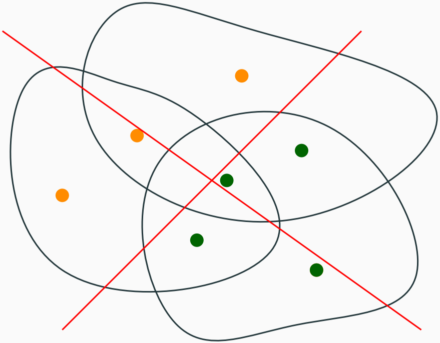

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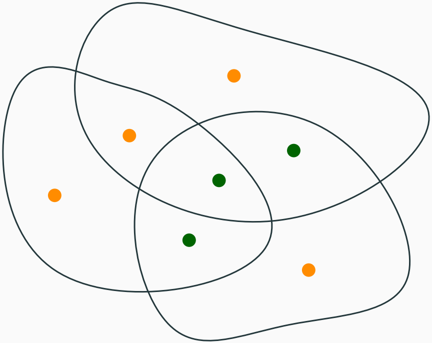

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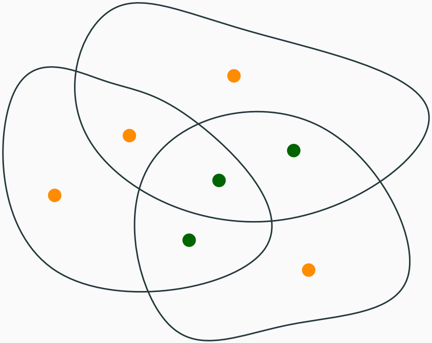

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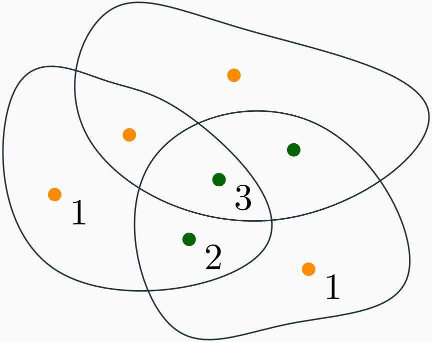

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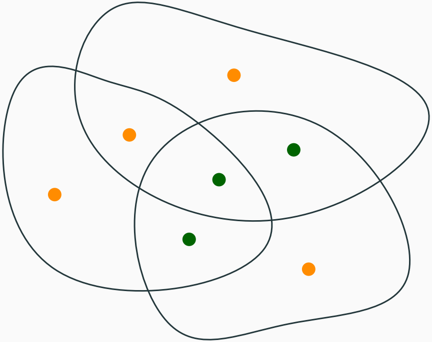

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An hypergraph if r-**regular** if every edge is of size r.

Hypergraph colorings: a result

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Theorem

Let $\beta \geq 1$. Let H be a r-uniform hypergraph of maximum degree Δ,

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Proof of the Theorem: By induction, $\mathcal{C}(H) \geq \beta^{|H|}$ $\square$

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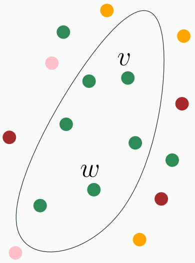

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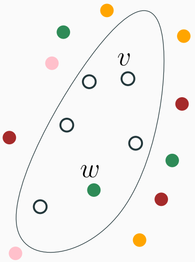

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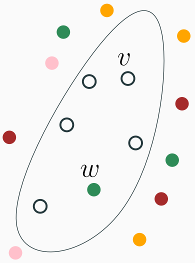

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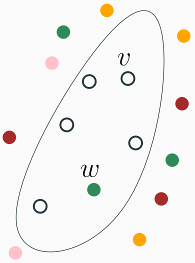

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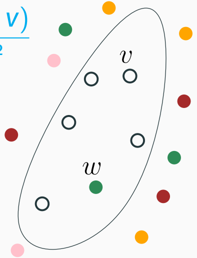

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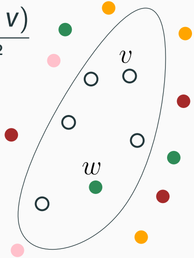

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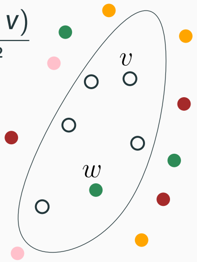

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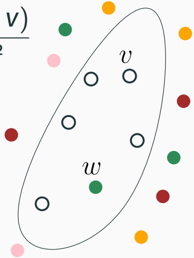

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$$\chi(H) \leq c := \min_{\beta > 0} \left\lceil \beta + \frac{\Delta}{\beta^{r-2}} \right\rceil .$$

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Let $r > 2$. *Let H be a r-uniform hypergraph of maximum degree Δ,*

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Remark: For the chromatic number of graphs (2-regular hypergraph), we have $c = \Delta + 1$.

A second example:
Star coloring

Star coloring: definition

A star coloring of a graph G is a proper coloring such that any pair of color classes induces a forest of stars.

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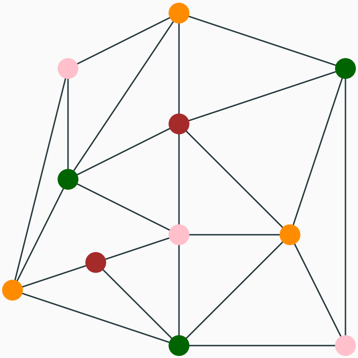

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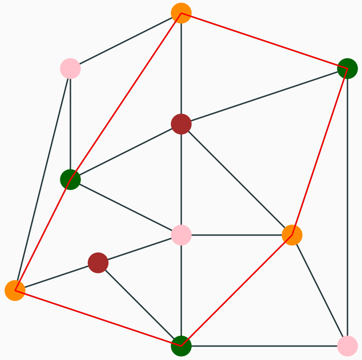

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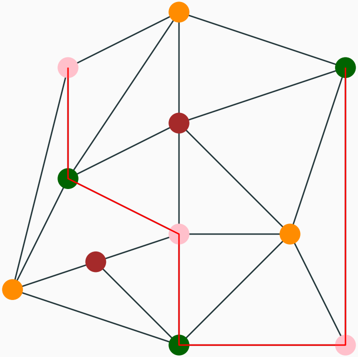

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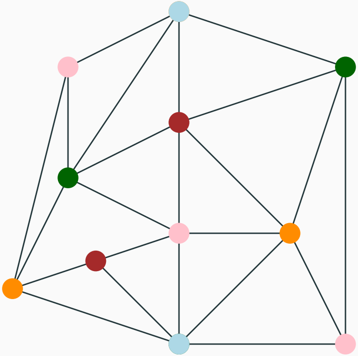

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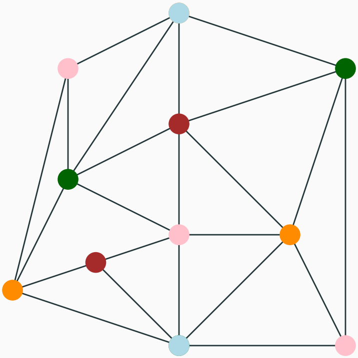

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Star coloring: results

For all graph G of maximum degree Δ,

Theorem (Fertin, Raspaud, and Reed, 2004 LLL)

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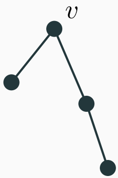

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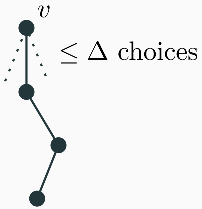

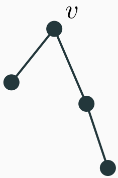

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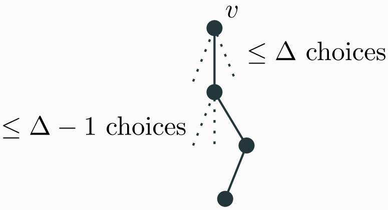

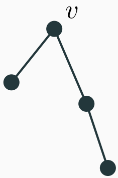

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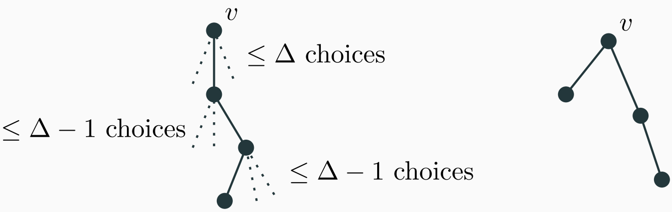

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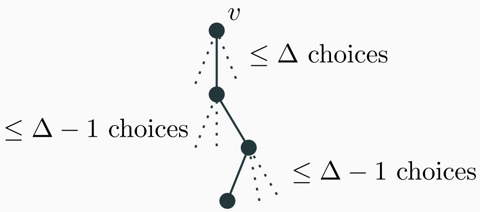

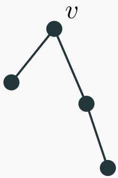

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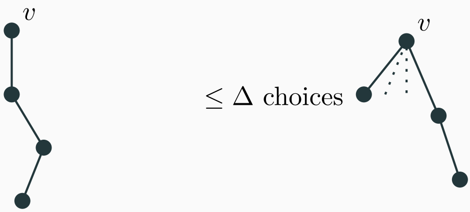

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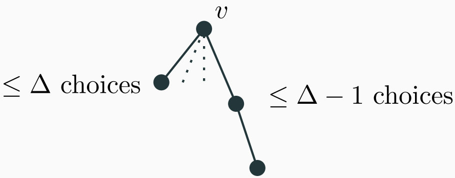

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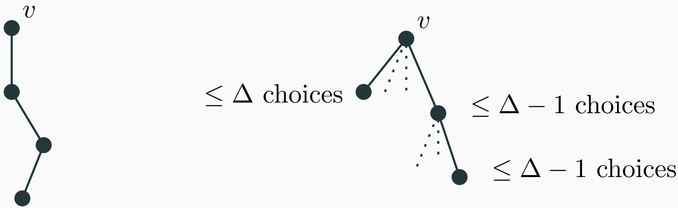

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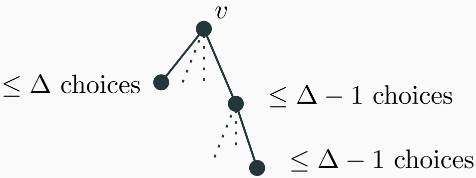

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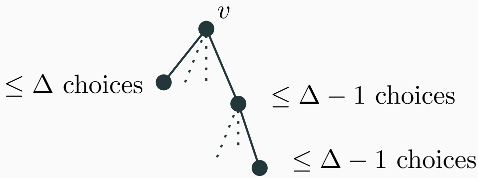

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Star coloring: with the counting argument

Theorem (This talk)

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Let $c = \left\lceil 2\sqrt{2}\Delta^{3/2} + \Delta \right\rceil$.

We let $\mathcal{C}_s(G)$ be the number of star c-colorings of G.

Let β such that $c - \Delta - \frac{2\Delta^3}{\beta} \geq \beta$.

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Theorem is a corollary of:

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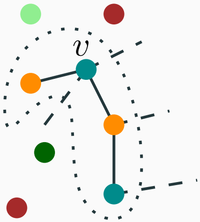

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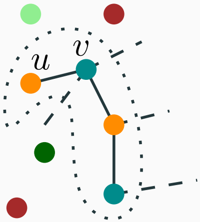

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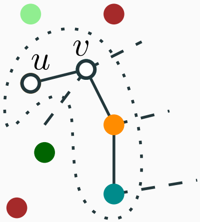

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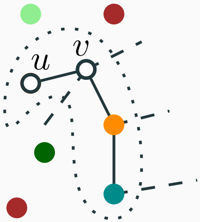

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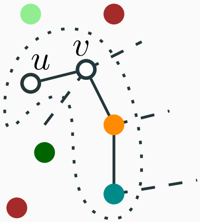

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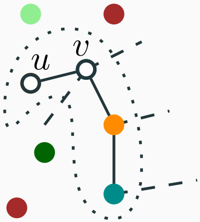

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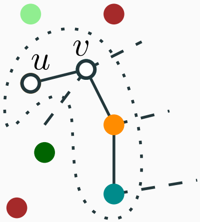

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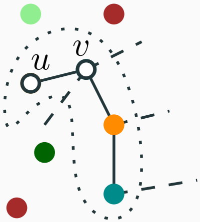

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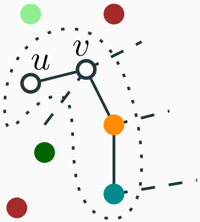

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Let G be a graph of maximum degree Δ, then

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Other applications

Chromatic number of triangle-free graphs of bounded degree

Problem (Vizing , 68)

If $\Delta(G)$ is the maximum degree of a vertex in a graph G, it is clear that $\chi(G) \leq \Delta(G) + 1$. [...] Perhaps one should start with estimates of the chromatic number of a graph without triangles ($\omega = 2$) and with given maximal degree for vertices.

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Similar problem already mentioned in the 50's by different authors (Erdős, Mycielski, Zykov...).

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Theorem (Johanson, 1996)

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Tight up to a factor 2

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Other results

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- group theory,
- ...

Wanless and Wood framework

Theorem (Wanless and Wood, 2020)

Let $(G, \mathcal{B})$ be an instance. Assume there exist a real number $\beta \geq 1$ and an integer $c \geq 1$ such that for every vertex v of G,

$$c \geq \beta + \sum_{k \geq 0} \beta^{-k} E_k(v) \, .$$

Then G is $(\mathcal{B}, c)$-choosable. Moreover, for every c-list assignment L of G,

$$P(G, \mathcal{B}, L) \geq \beta^{|V(G)|} \, .$$

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Try to apply it to your favorite problem =)

Thanks !