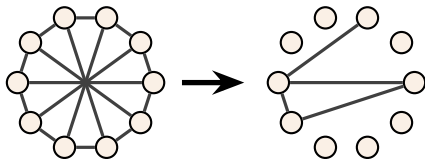


k -vertex-minor universal graphs and small k -pairable quantum states

Nathan Claudet, Mehdi Mhalla, Simon Perdrix

Journées Graphes et Algorithmes 2023 - 22/11/23
arXiv:2309.09956

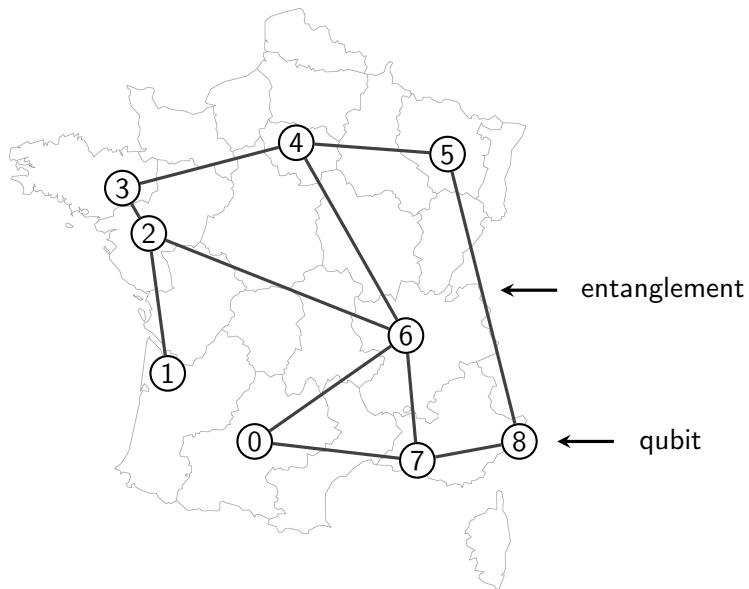


Outline

- 1 Motivation - quantum communication networks
- 2 k -vertex-minor universal graphs
- 3 Some results
- 4 A variation : k -pairable graphs
- 5 Conclusion

Motivation - quantum communication networks

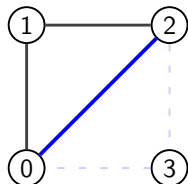
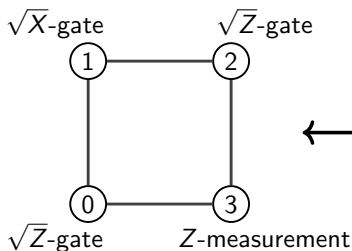
Quantum communication networks



Correspondence between quantum graph states and graphs

$|G\rangle$

G



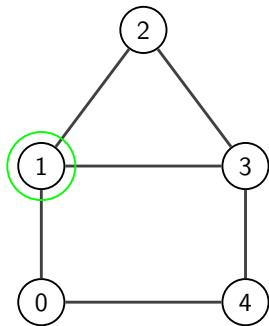
local (i.e. single-qubit)
quantum operations

local complementations
& vertex disconnections

Local complementation

Definition (Local complementation)

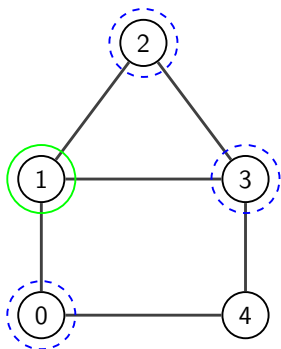
Given a graph G , a local complementation on a vertex u consists in complementing the (open) neighborhood of u in G .



Local complementation

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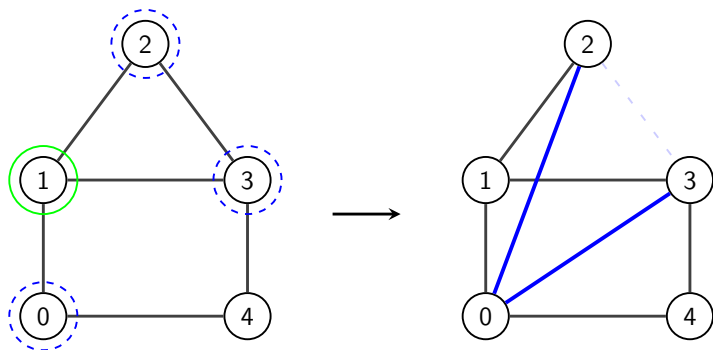
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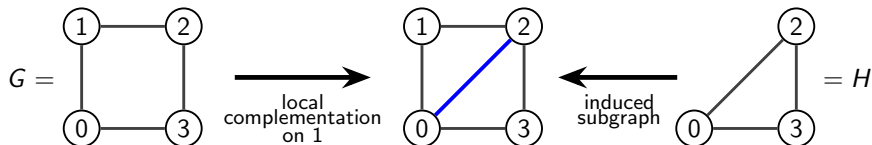


k -vertex-minor universal graphs

Vertex-minors

Definition (Vertex-minor)

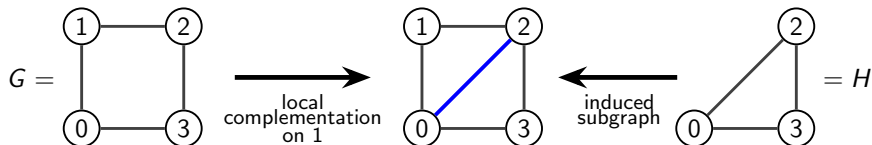
Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ such that $V_H \subseteq V_G$, H is a vertex-minor of G if H can be obtained as a induced subgraph of G by means of local complementations.



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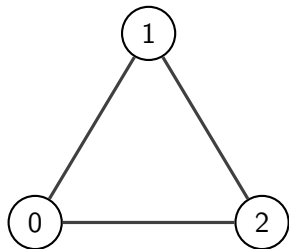


Definition

A graph G is k -vertex-minor universal if any graph on any k vertices is a vertex-minor of G .

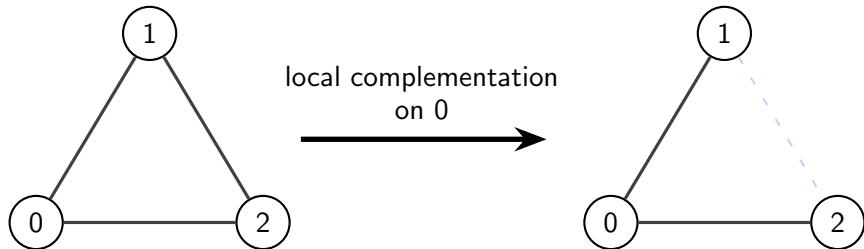
k -vertex-minor universal graphs : example 1

K_3 is 2-vertex-minor universal.



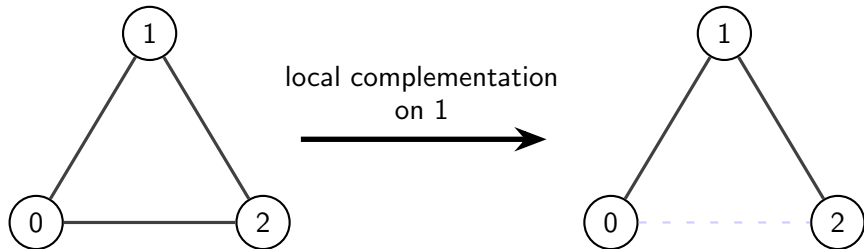
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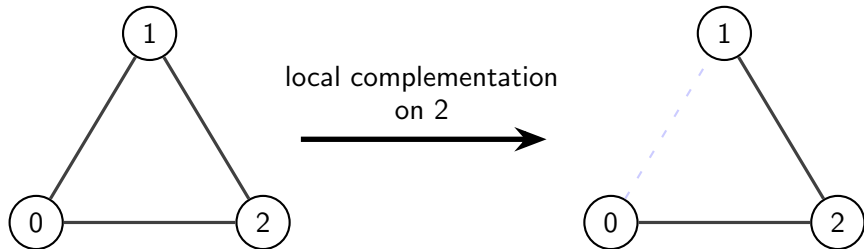
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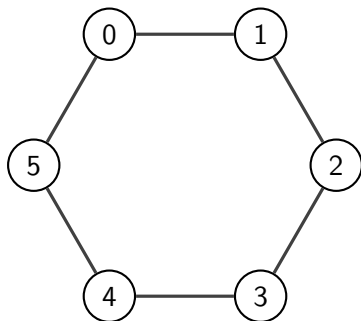
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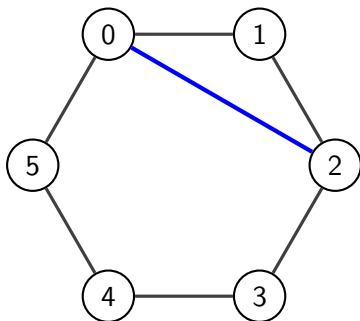
k -vertex-minors-universal graphs : example 2

C_6 is 3-vertex-minor universal.



k -vertex-minors-universal graphs : example 2

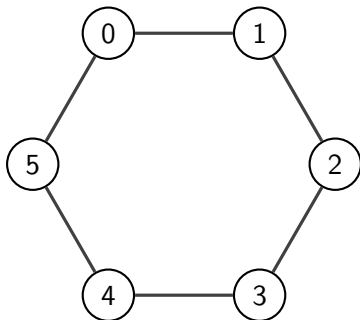
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To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.

k -vertex-minors-universal graphs : example 2

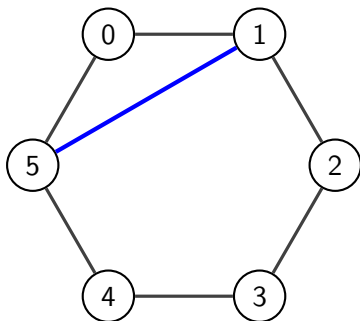
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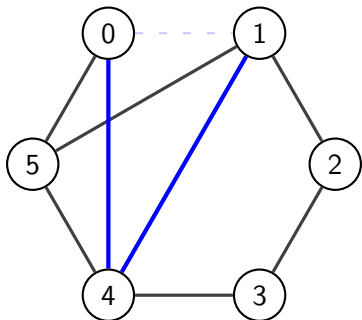
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To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
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k -vertex-minors-universal graphs : example 2

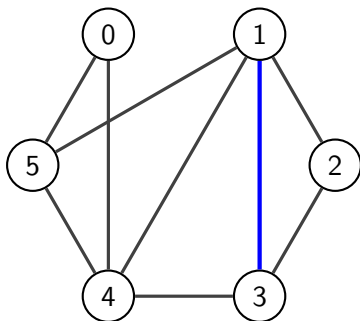
C_6 is 3-vertex-minor universal.



To induce the complete graph on $\{0, 1, 2\}$: Local complementation on 1.
To induce the empty graph on $\{0, 1, 2\}$: Local complementation on 0, on 5,

k -vertex-minors-universal graphs : example 2

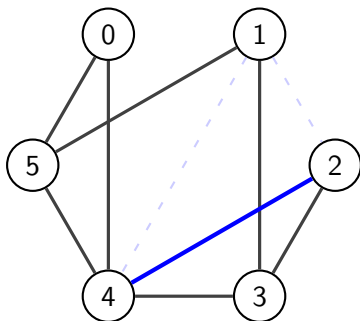
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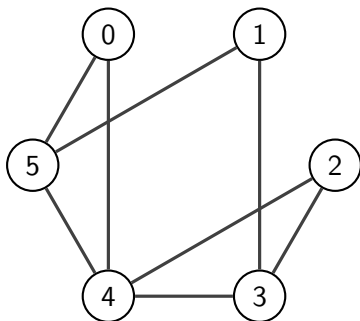
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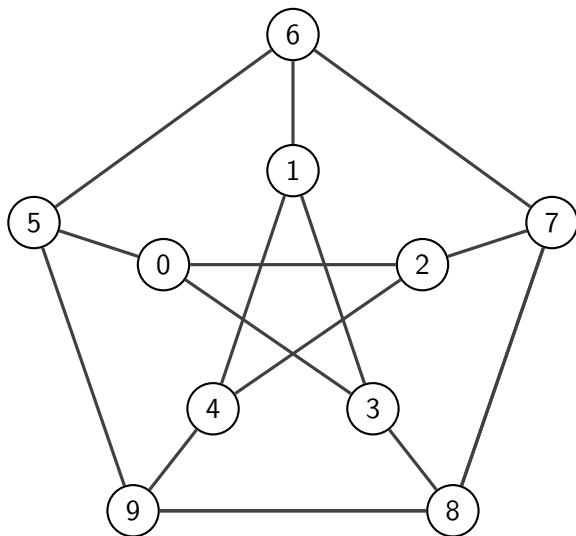
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k -pairable states : example 3

The Petersen graph is 4-vertex-minor universal.



Some results

Polynomial-size k -vertex-minors-universal graphs

Proposition (Claudet, Mhalla, Perdrix 2023)

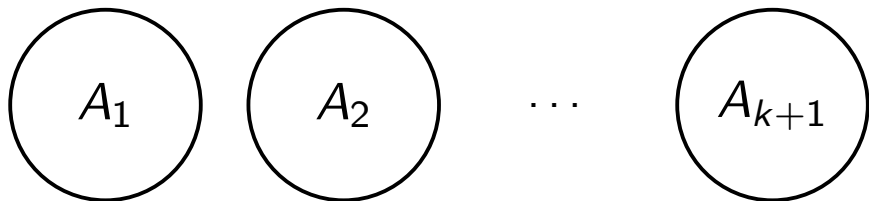
For any k , there exists a k -vertex-minor universal graph of order $n = O(k^4 \ln(k))$.

Polynomial-size k -vertex-minors-universal graphs

Proposition (Claudet, Mhalla, Perdrix 2023)

For any k , there exists a k -vertex-minor universal graph of order $n = O(k^4 \ln(k))$.

Idea of proof: Probabilistic construction of a balanced $(k + 1)$ -partite graph.



Probability of two vertices u and v sharing an edge : 0 if u and v are in the same component, $2/k$ else. $\forall K \in \binom{V}{k}, \exists i$ s.t. $K \cap A_i = \emptyset$.

An upper bound on k -vertex-minor universality

Proposition

For any k , a k -vertex-minor universal graph is at least of order $n = O(k^2)$.

A variation : k -pairable graphs

Definition

A graph G is k -pairable if any perfect matching on any $2k$ vertices is a vertex-minor of G .

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$2k$ -vertex-minor universality $\implies k$ -pairability.

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Proposition

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Proposition (Claudet, Mhalla, Perdrix 2023)

For any k , there exists a k -pairable graph on $n = O(k^3 \ln^3(k))$ qubits.

Conclusion

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Introduction of the notion of k -vertex-minor universal graphs.

Main result: for any k there exist k -vertex-minor universal graph of order $O(k^4 \ln(k))$.

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Future work:

- Explicit constructions?
- Existence of k -vertex-minor universal graphs of order $O(k^3)$? $O(k^2)$?

Conclusion

Introduction of the notion of k -vertex-minor universal graphs.

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Future work:

- Explicit constructions?
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- k -pairability = $2k$ -vertex-minor universality?

Thanks



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