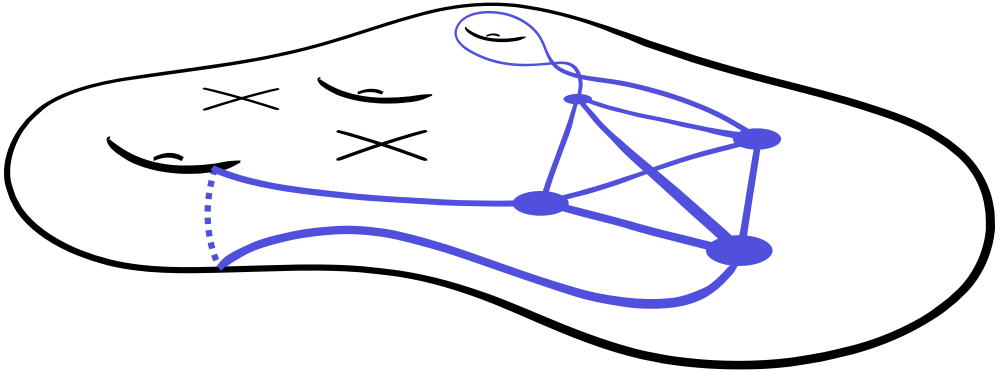
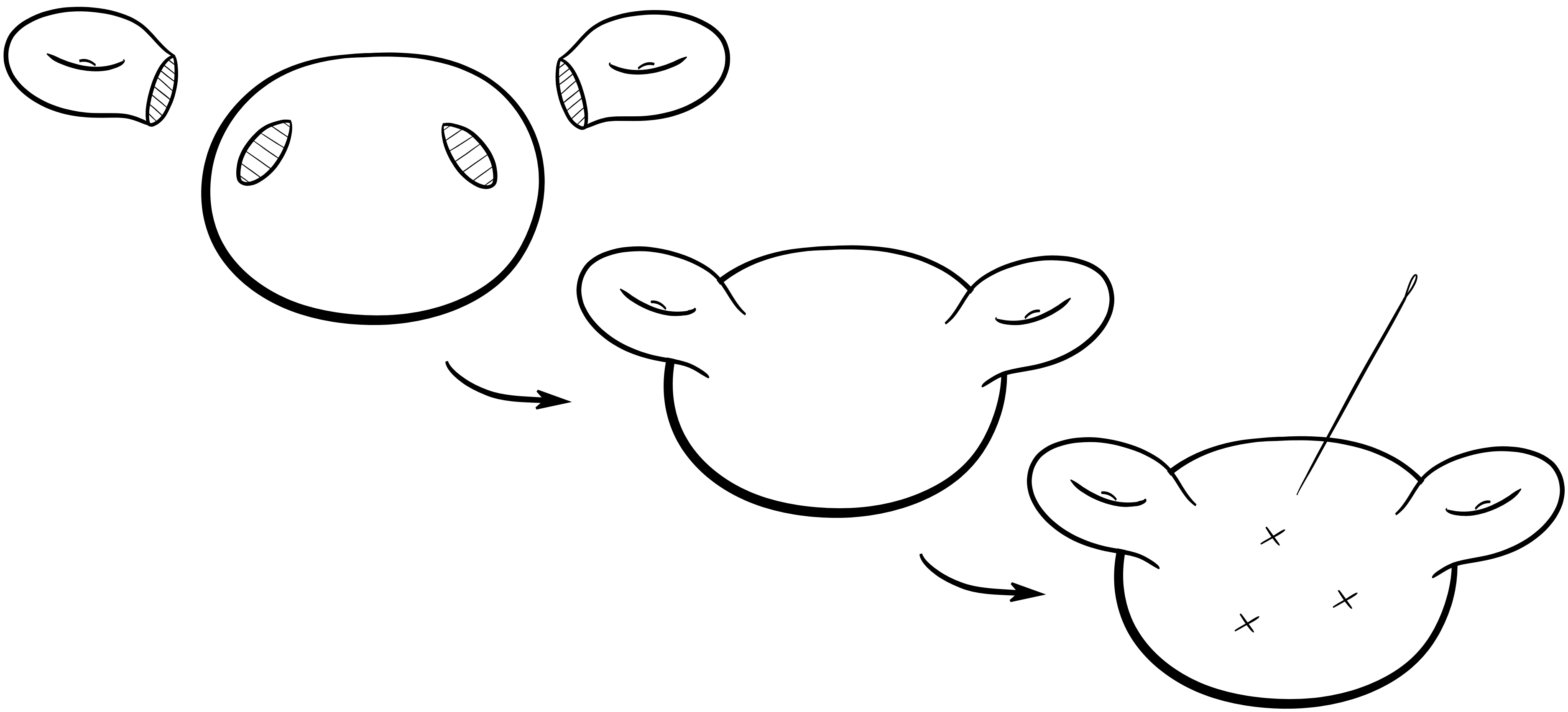
Untangling Graphs on Surfaces



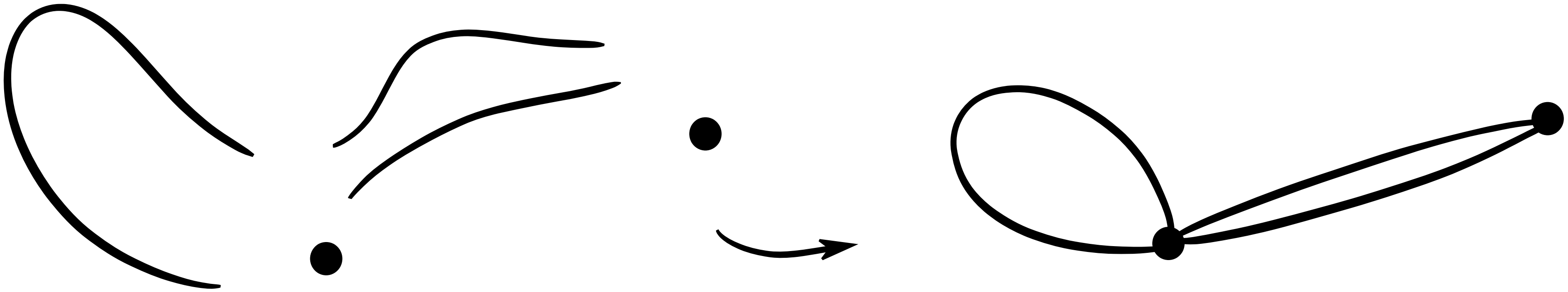
Éric Colin de Verdière
Vincent Despré
Loïc Dubois

SODA'24
arXiv:2311.00437

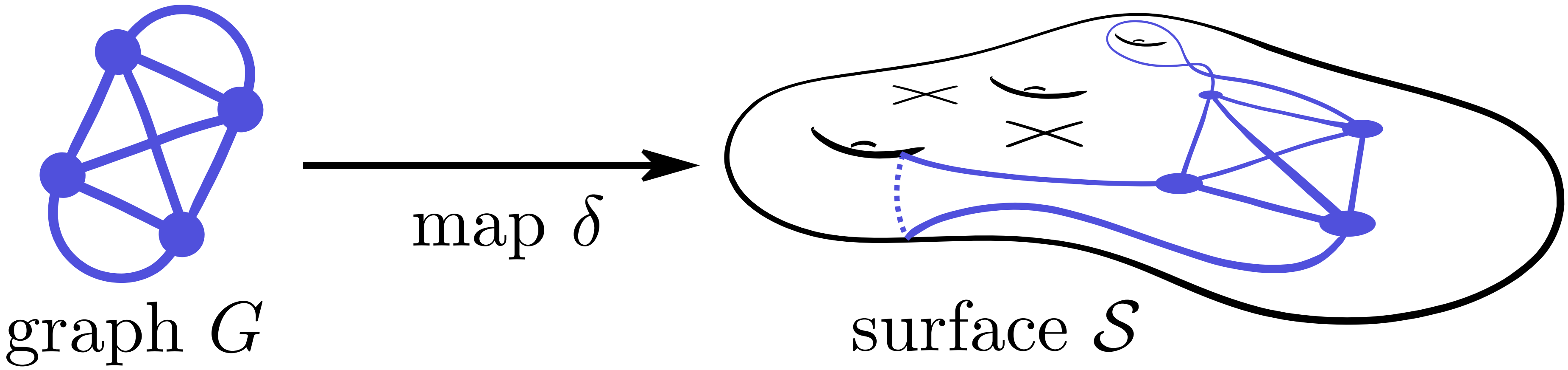
(topological) surface $\mathcal{S}$:



(topological) graph G :

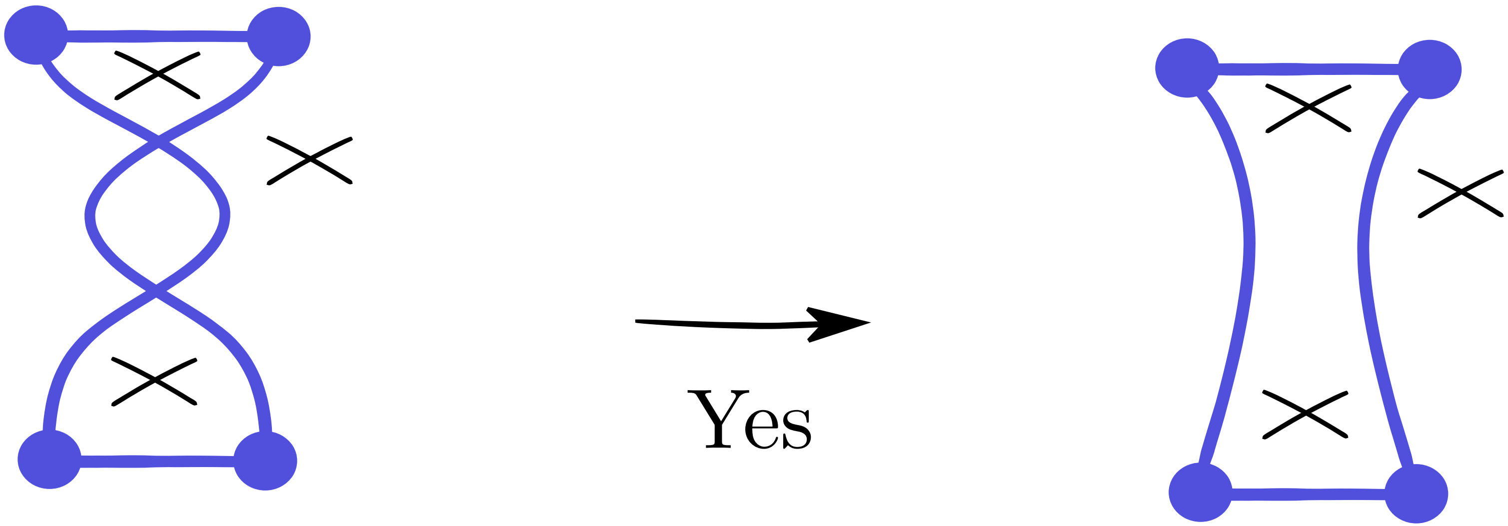


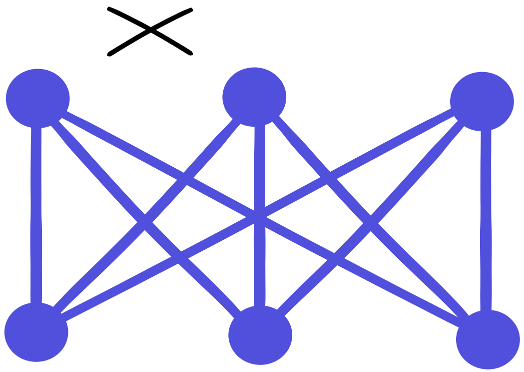
The problem



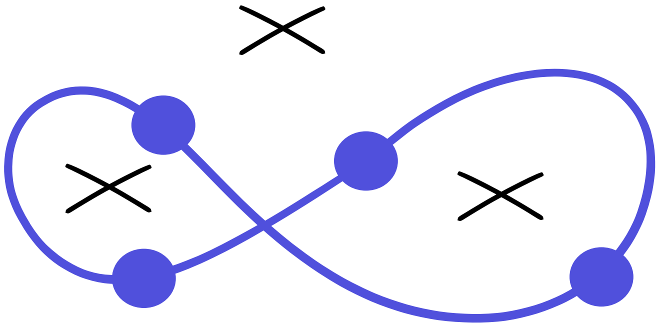
can δ be untangled by continuous motion?

our result : polynomial time algorithm


Yes



No



No

Related works

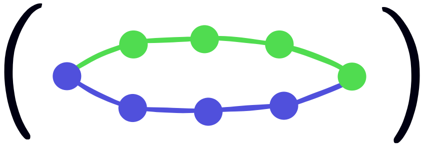
Mohar, 1999 : $\forall$ surface $\exists$ linear time algorithm to decide if a graph can be embedded on the surface

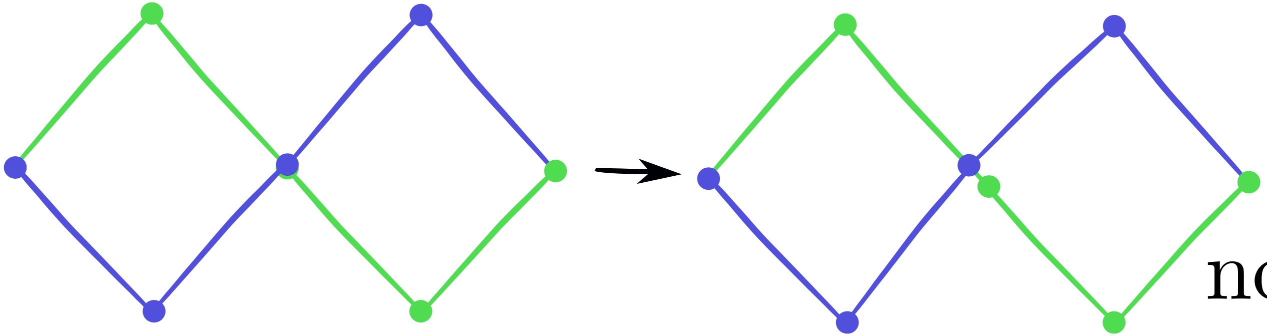
not our problem !

Despré, Lazarus, 2019 : $\exists$ quasi-linear time algorithm to decide if two curves can be untangled on a surface

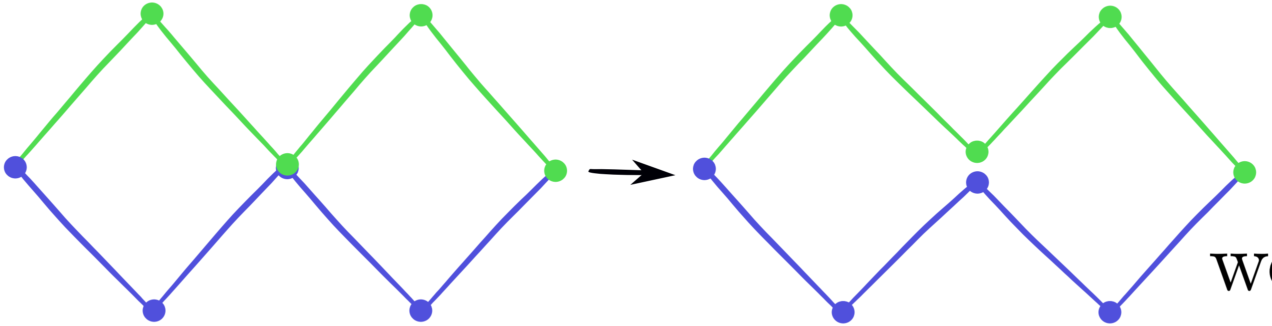
we generalize to graphs

Related works





not weak-embedding



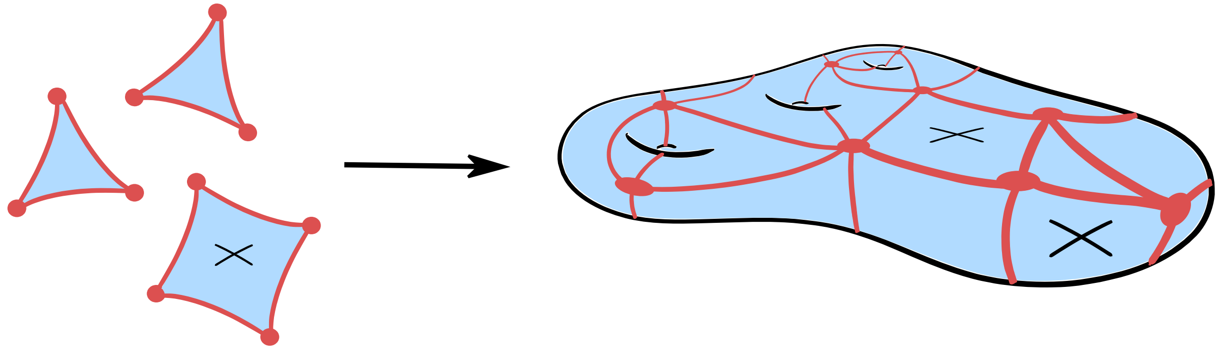
weak-embedding

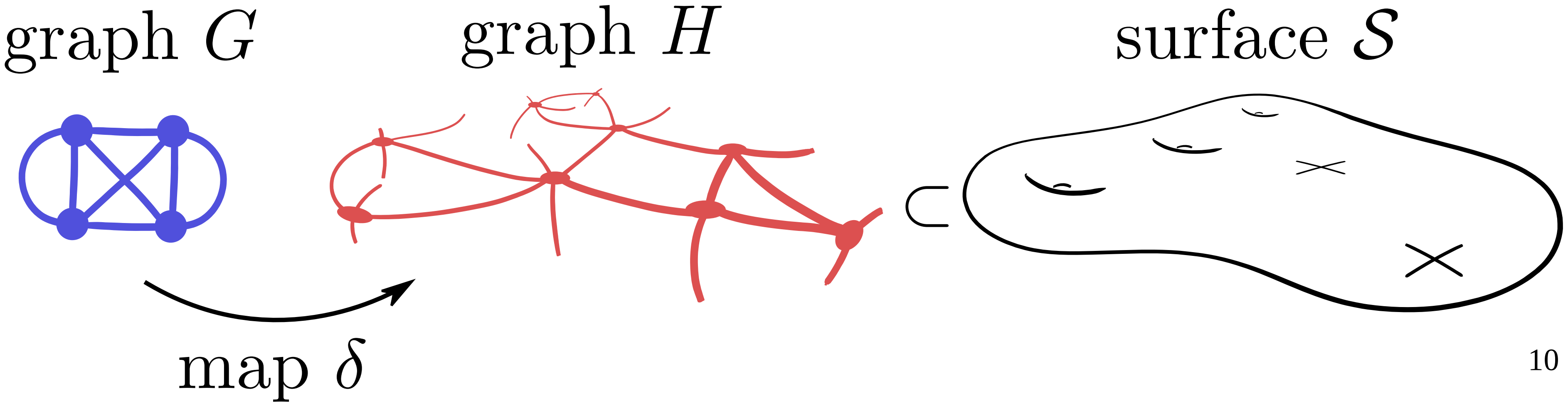
Related works

Akitaya, Fulek, Tóth, 2019 : $\exists$ quasi-linear time algorithm to recognize weak-embeddings

↪ key subroutine for us

Why the input is discrete





Our result

input : $\delta : G \to H \subset \mathcal{S}$

m : # vertices and edges of H

n : # times δ uses an edge or vertex of H

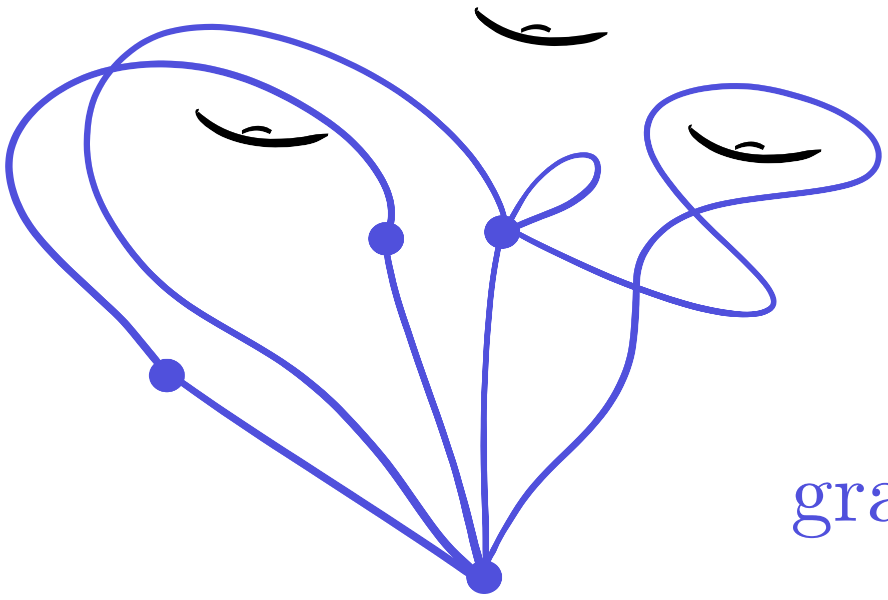
s : # obstacles and handles in $\mathcal{S}$

Colin de Verdière, Despré, D., 2023 : We can decide if δ can be untangled, in $O(m + s^2 n \log(sn))$ time

(If so, we can construct an untangled version of δ)

Overall algorithm

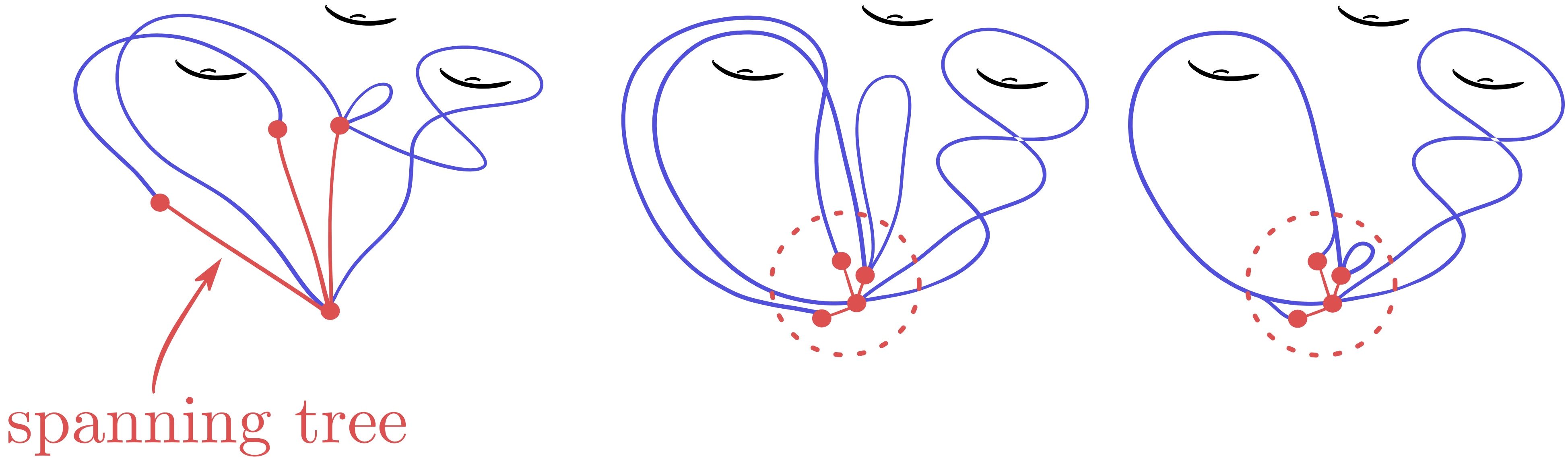
Overall algorithm



graph G

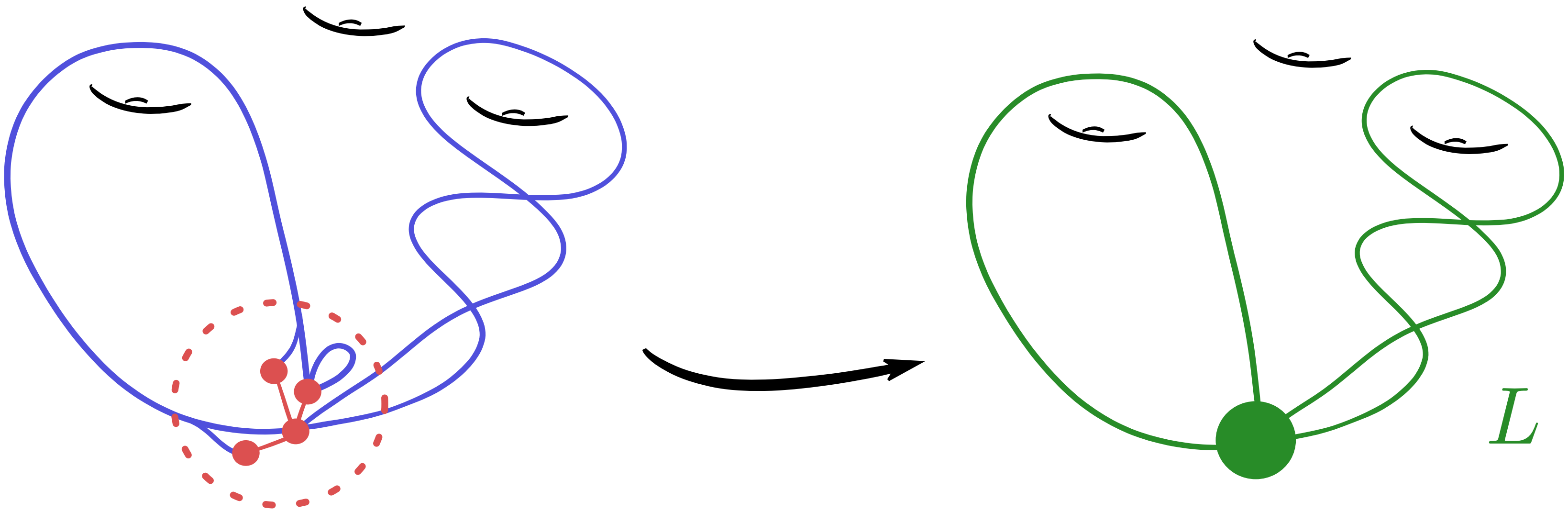
Overall algorithm

1. Contract and bundle homotopic loops



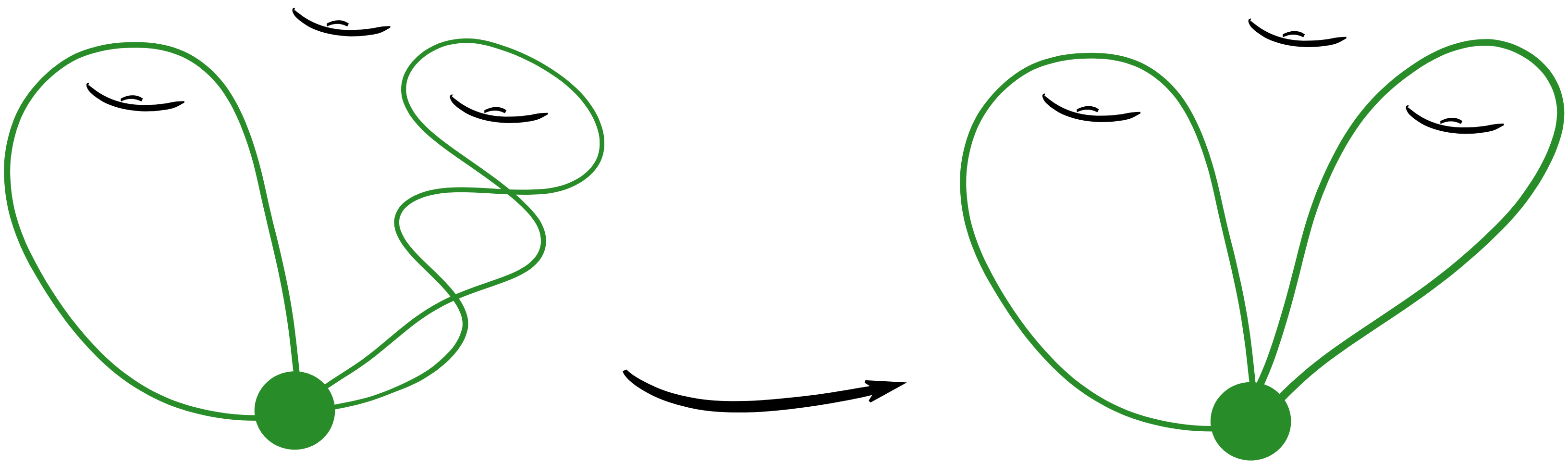
Overall algorithm

now G is in the neighborhood of a one-vertex graph L



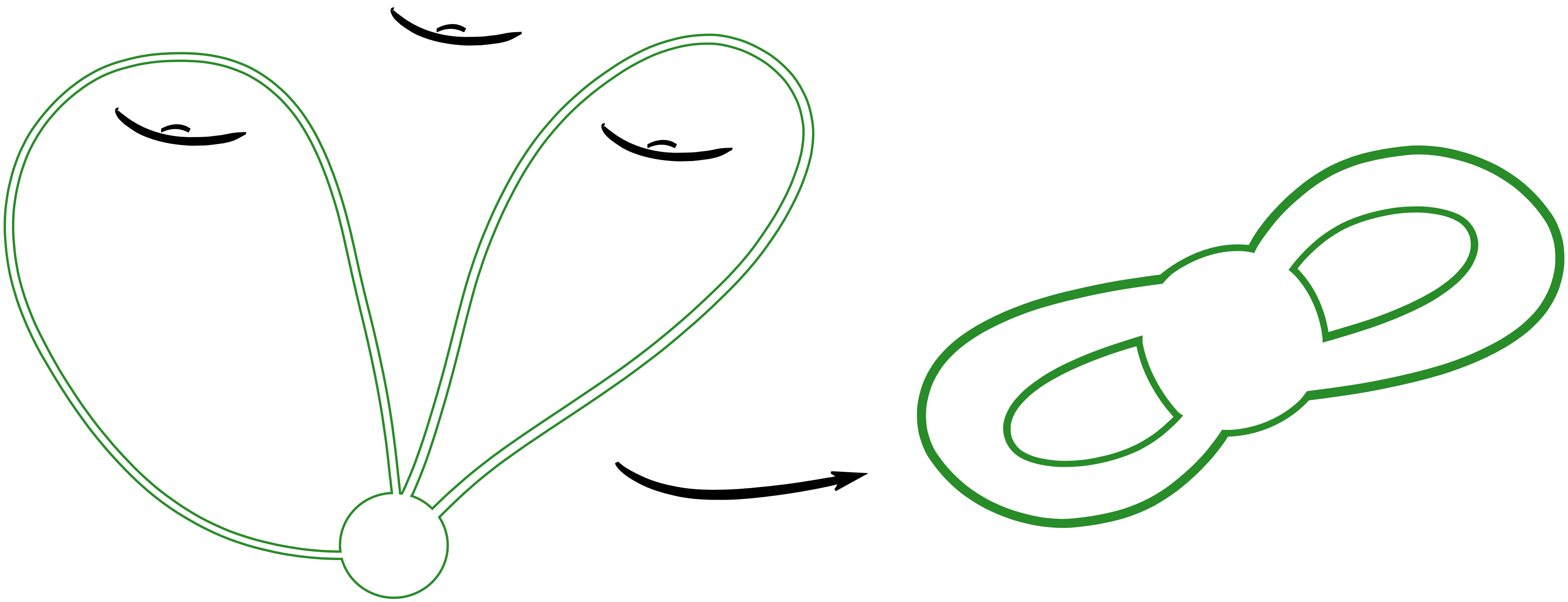
Overall algorithm

2. Prove G can be untangled $\Rightarrow$ L can be untangled
3. Untangle L (or return)

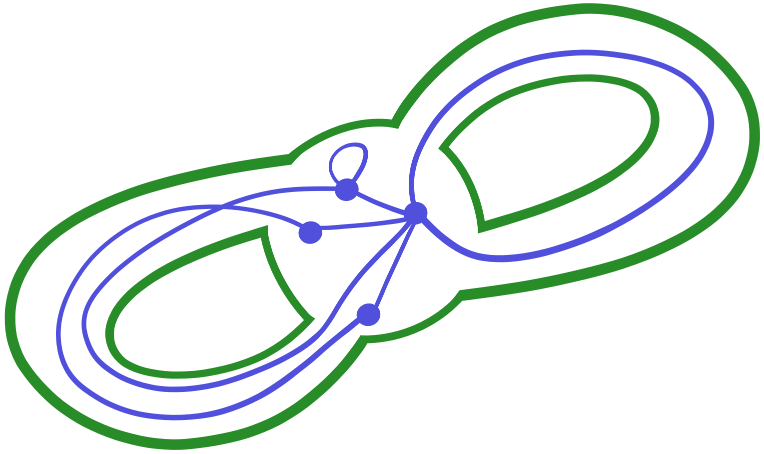


Overall algorithm

consider the neighborhood of L untangled



Overall algorithm

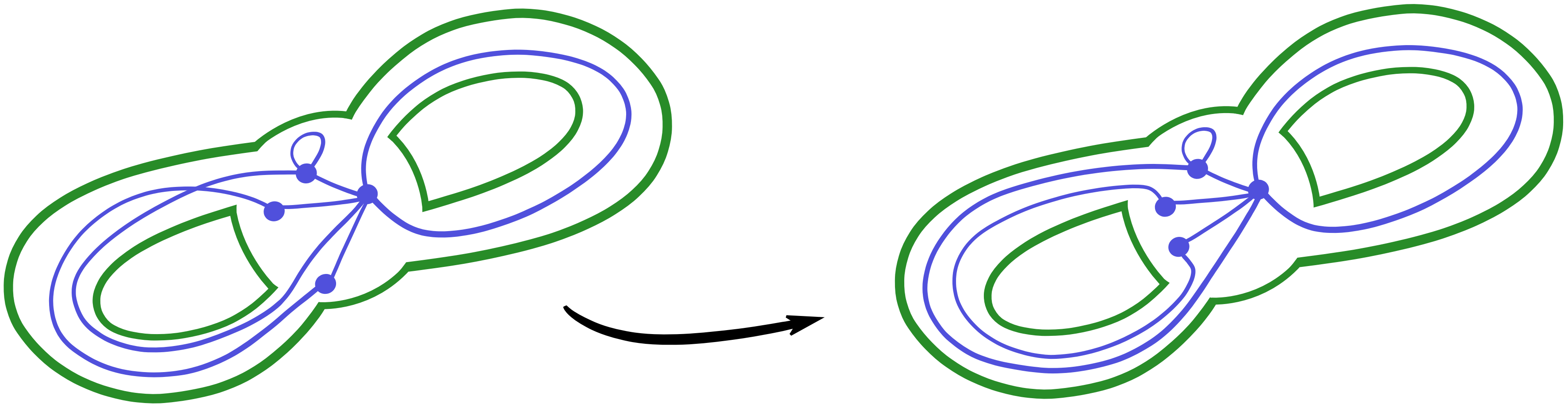


G is drawn in this neighborhood

Overall algorithm

4. Prove G can be untangled $\Rightarrow$ G is weak-embedding

5. Weak-embed G (or return)



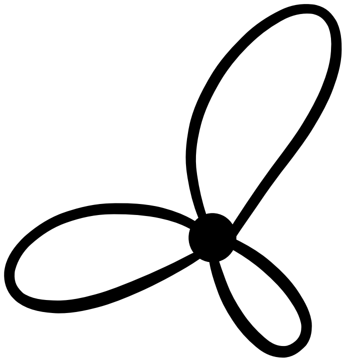
Summary

1. Contract and bundle homotopic loops $\rightarrow$ Tailored data structures
2. Prove G can be untangled $\Rightarrow$ L can be untangled
3. Untangle L (or return) $\rightarrow$ The core of the paper
4. Prove G can be untangled $\Rightarrow$ G is weak-embedding
5. Weak-embed G (or return)

$\rightarrow$ "Recognizing Weak Embeddings of Graphs"
Akitaya, Fulek, Tóth, 2019

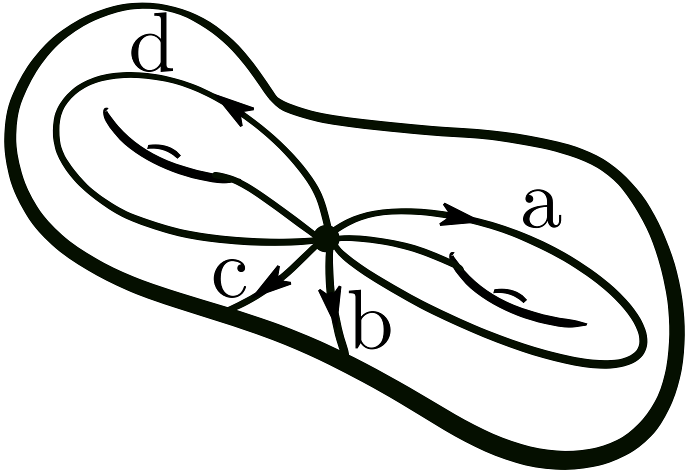
3. Untangle L (or return) $\rightarrow$ The core of the paper

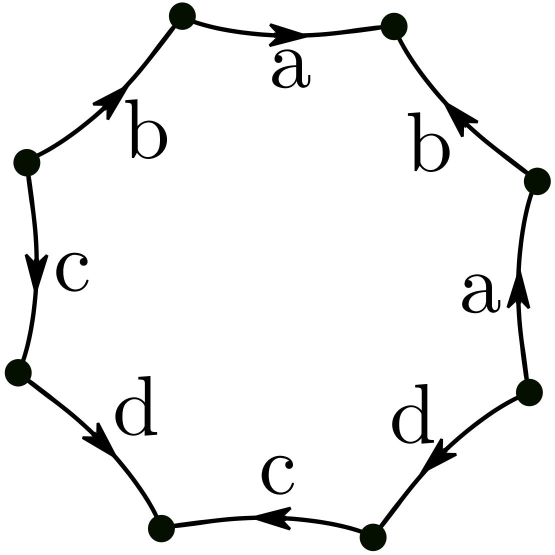
Untangling a loop graph

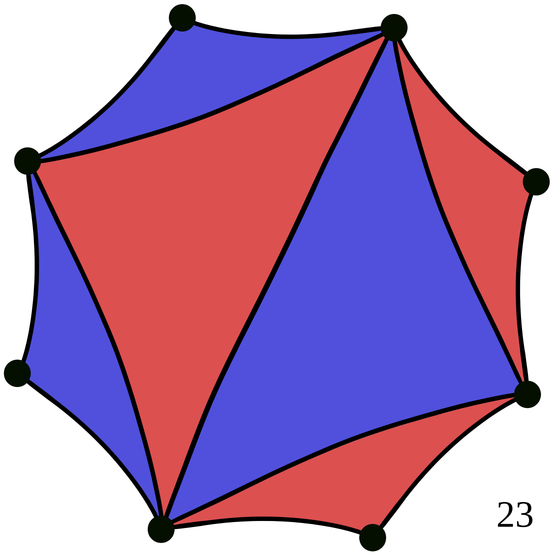


Reducing triangulations

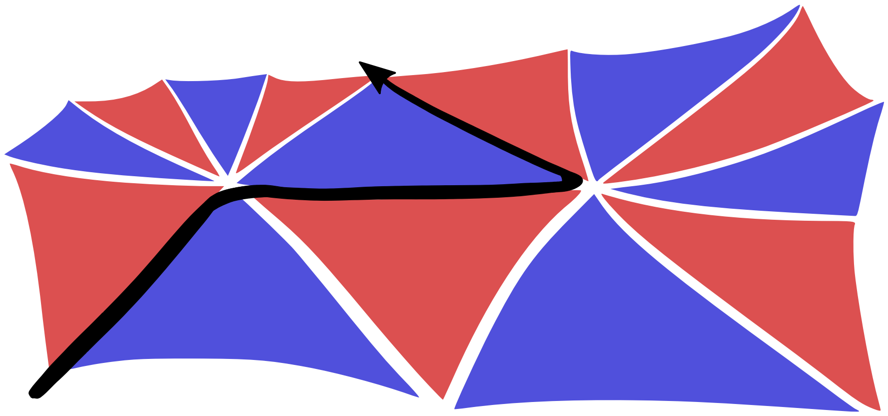
(def) reducing triangulation : vertices have degree ≥ 8
and dual is bipartite



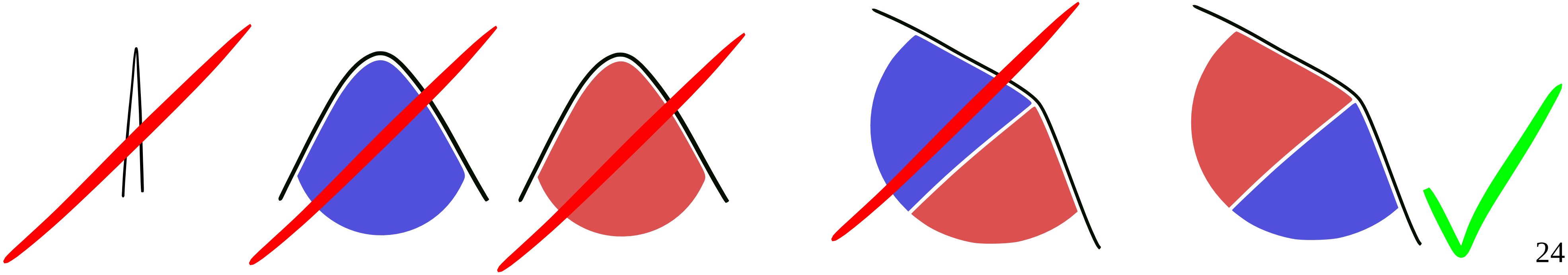




Reducing triangulations



(def) reduced walk : no bad turn

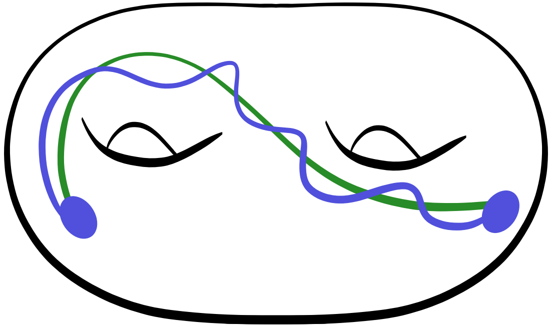


Reducing triangulations

Reduced walks are stable upon
reversal and subwalk and . . .

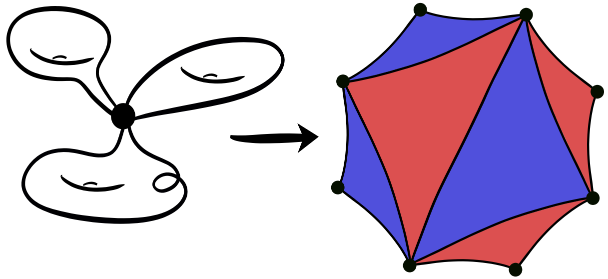
Theorem

> are unique in their homotopy class
> and can be computed in linear time

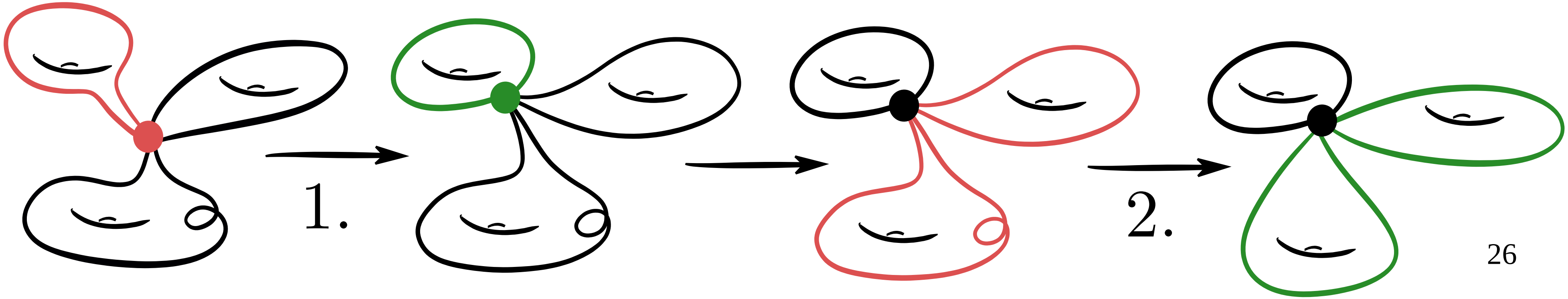


Straightening a one-vertex graph

0. Draw in a reducing triangulation



1. Make 1 loop reduced, moving the vertex
2. Make the other loops reduced, vertex fixed



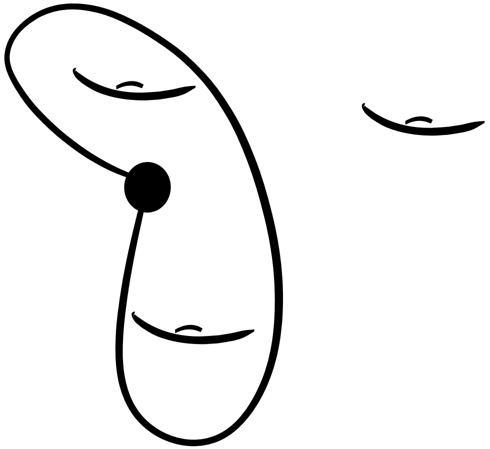
The key property

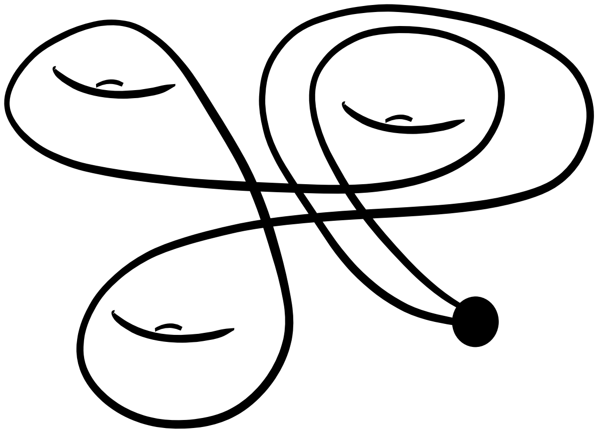
In a reducing triangulation . . .

Theorem

If a straightened one-vertex graph can be untangled, then it is a weak-embedding

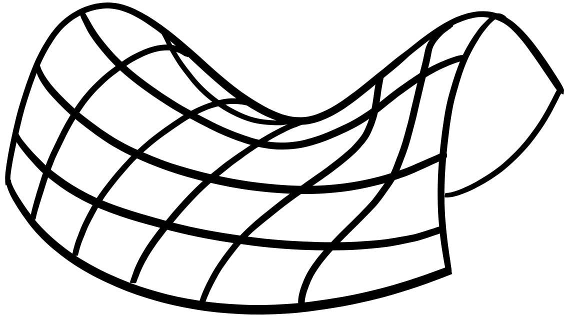
Why we first reduce cyclically one loop before reducing the other loops





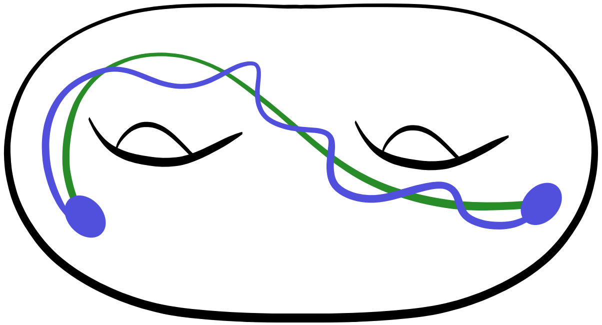
About the techniques we employ

Hyperbolic geometry

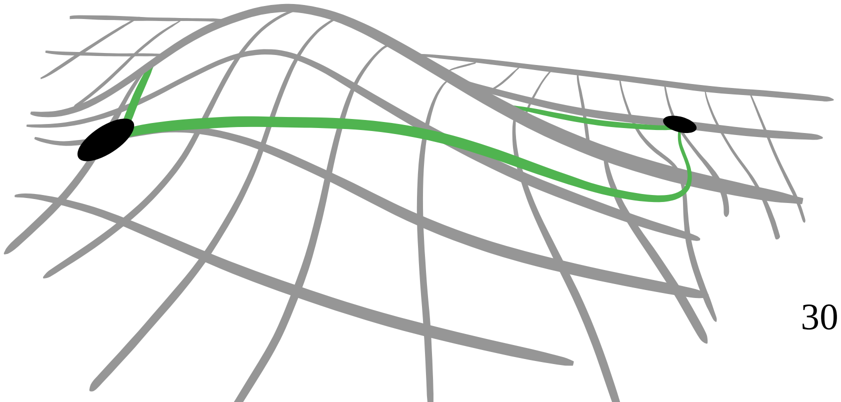


Hyperbolic surfaces

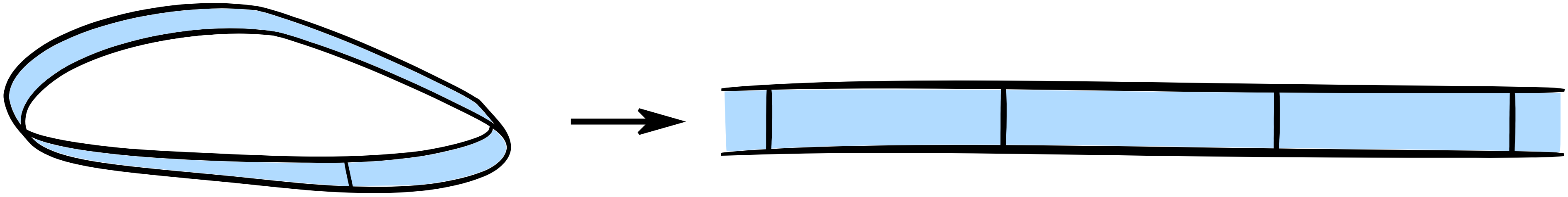
1 geodesic per homotopy class

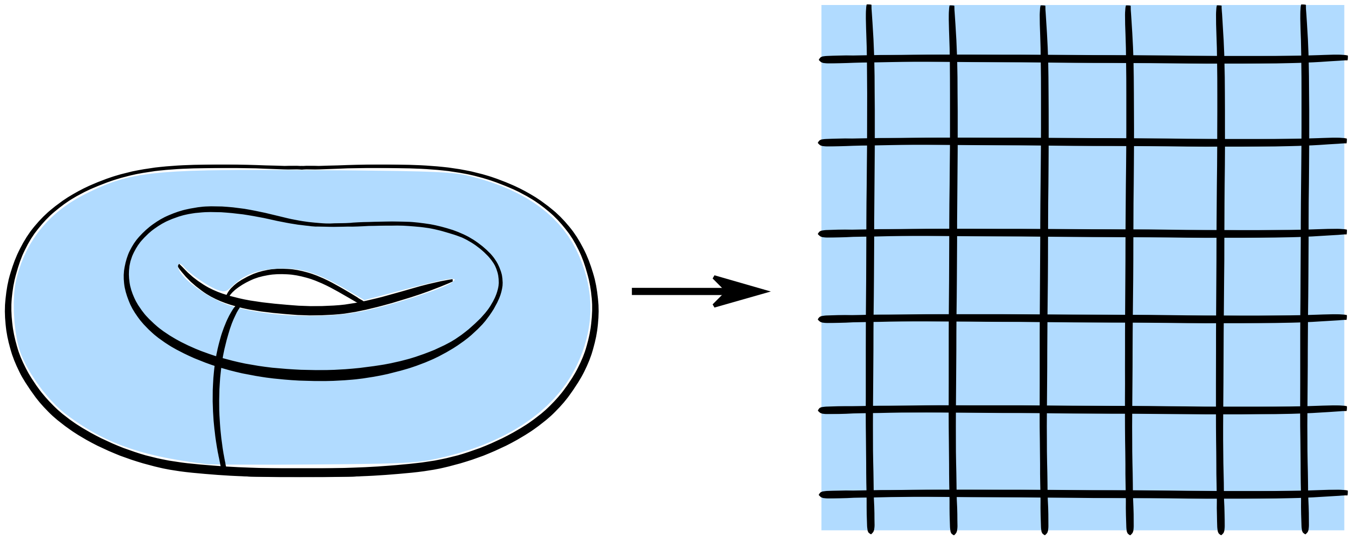


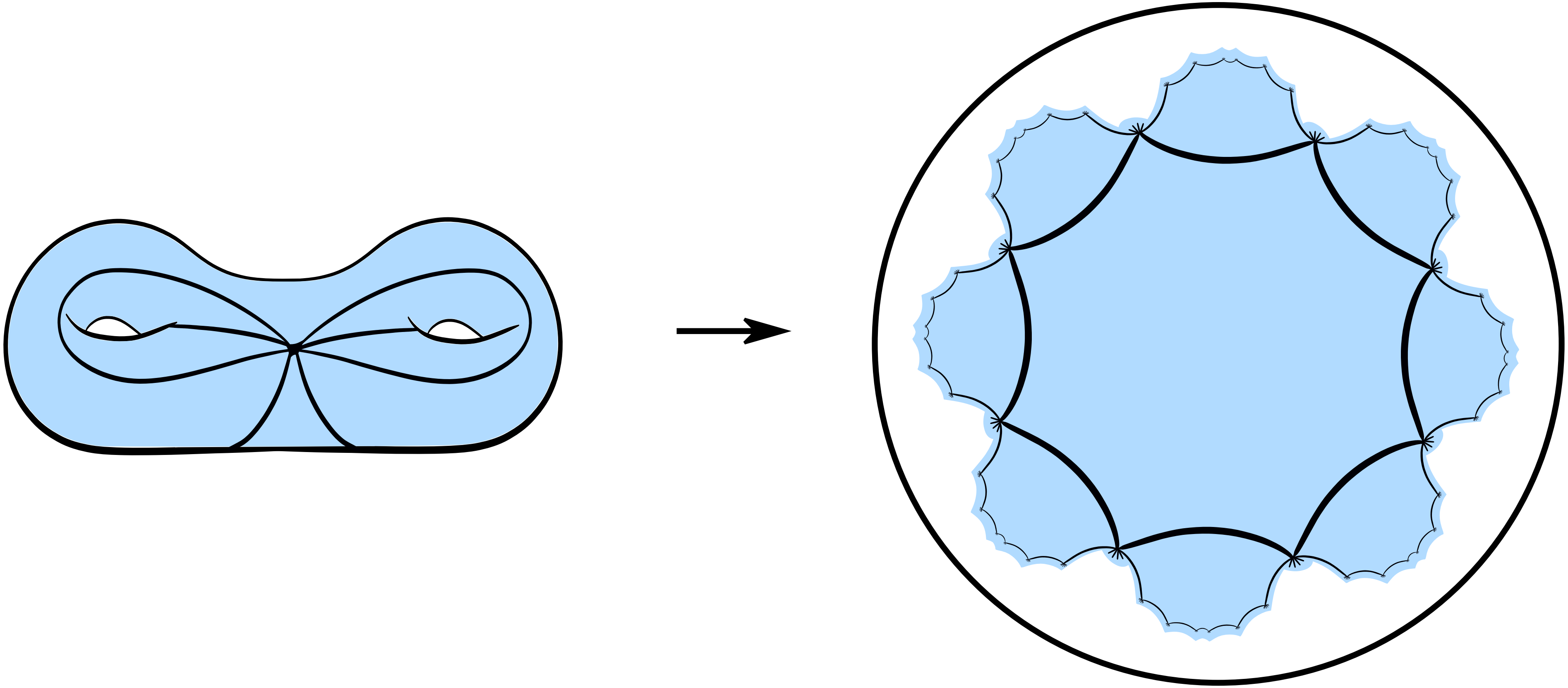
(not true for general surfaces)

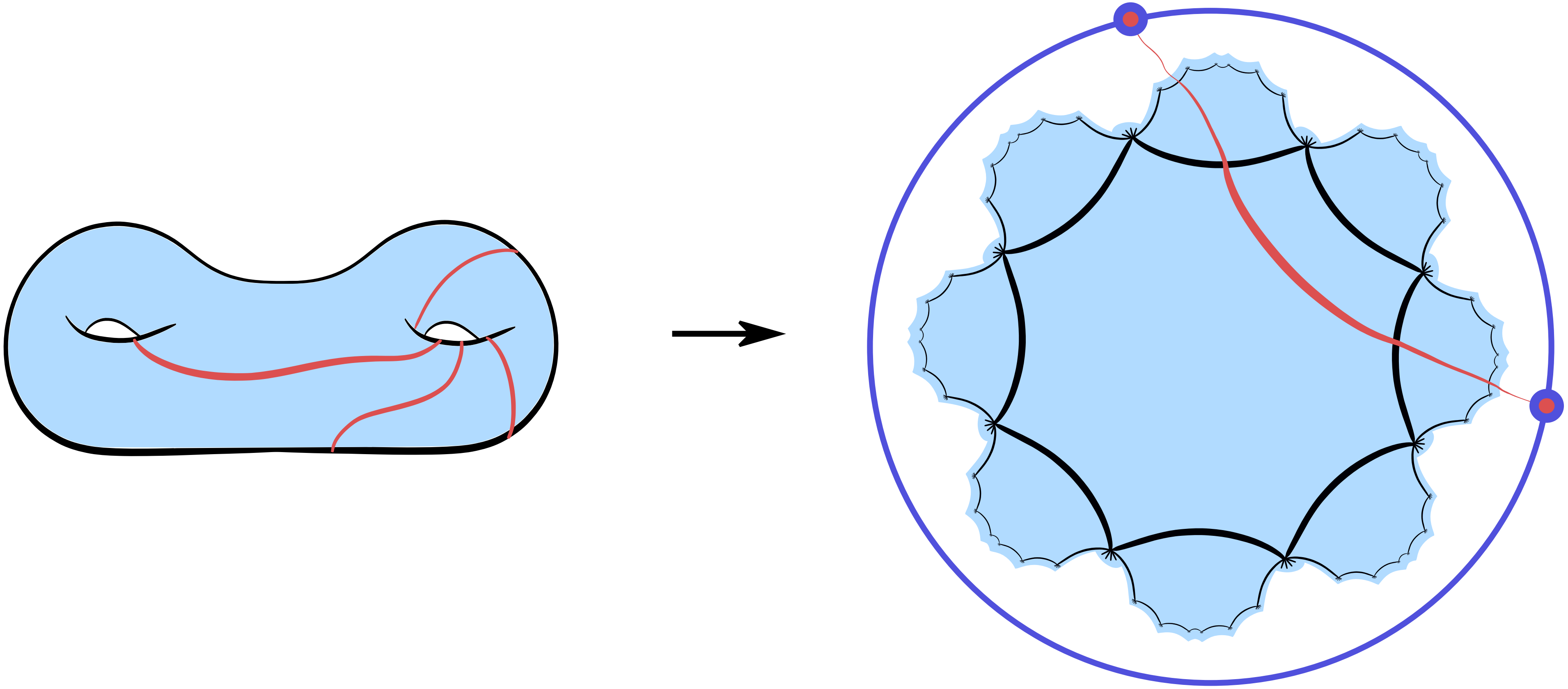


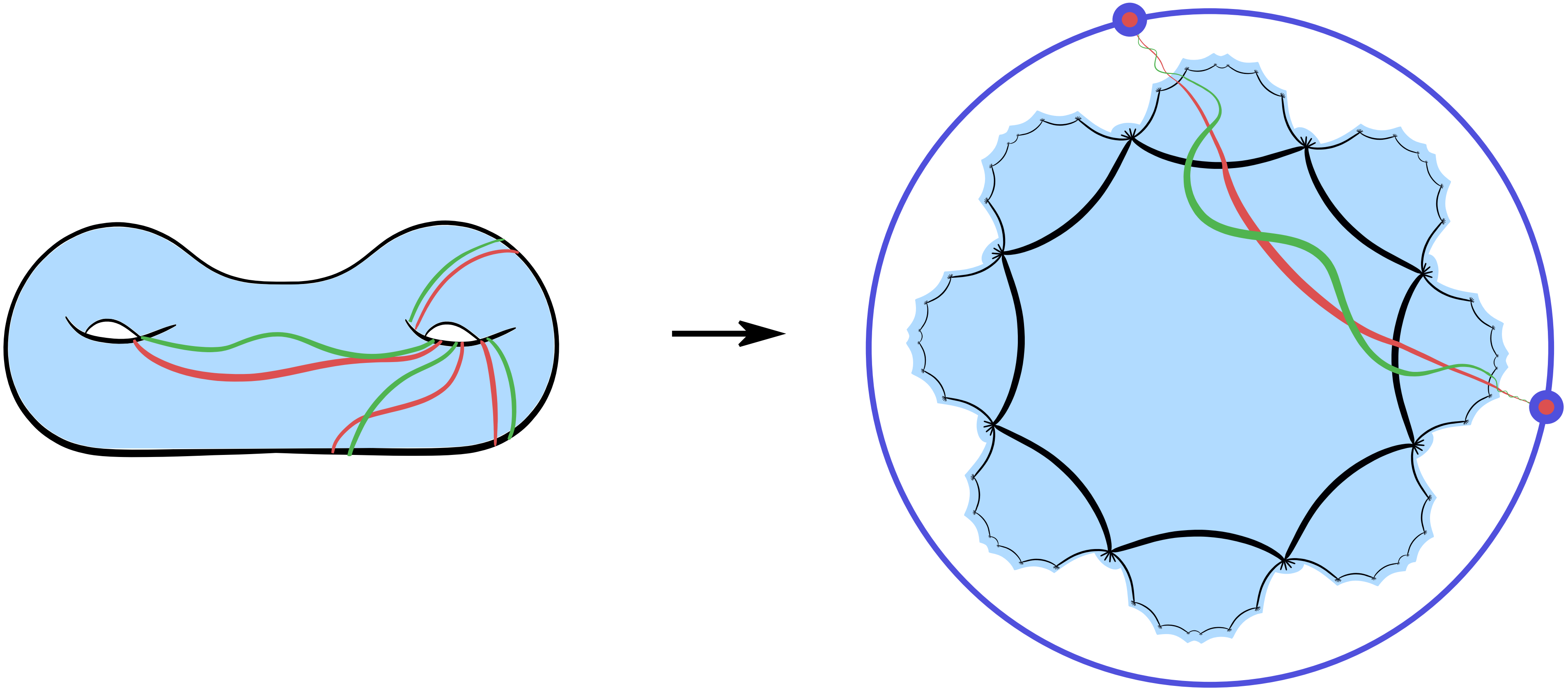
Covering spaces

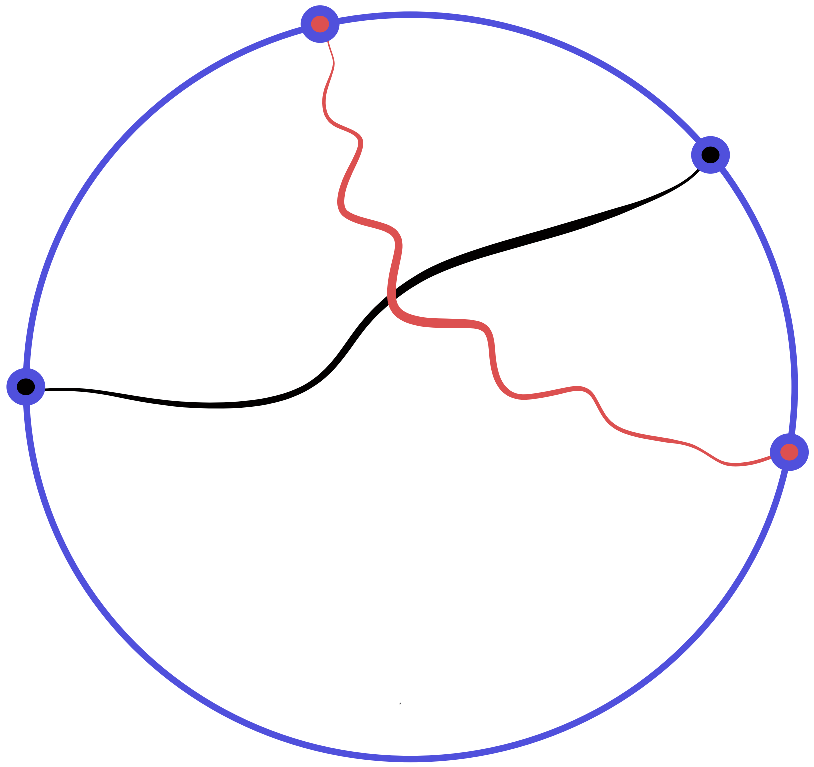






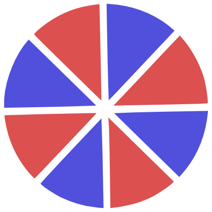
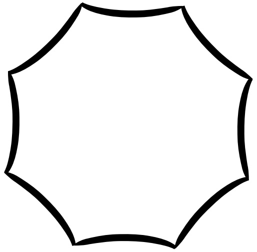
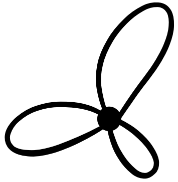






No continuous deformation

can untangle the red and black curves

Thank you