A note on interval colourings of graphs

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Joint work with M. Axenovich, A. Girão, L. Hollom, E. Powierski, M. Savery, Y. Tamitegama and L. Versteegen

What is an interval colouring?

Definition

An interval colouring of a graph $G = (V, E)$ is a proper edge-colouring $c : E \to \mathbb{Z}$ such that, for any vertex $v \in V$, the set of colours of edges incident to v is an interval of $\mathbb{Z}$.

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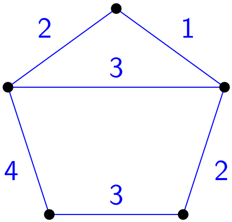

Figure: Example of interval colouring

Why are interval colourings important?

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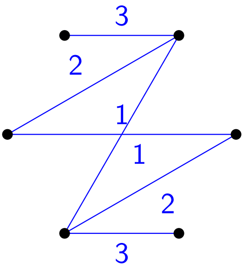

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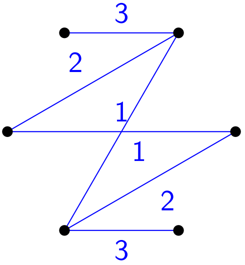
- Vertices are people, edges are meetings, and colours are times of meetings
- Bipartite graphs receive special attention; the ‘parent-teacher conference’ analogy often used



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The interval thickness of a graph $G = (V, E)$, written $\theta(G)$, is the minimum k such that G can be edge-partitioned into k parts, each of which can be interval coloured.

We also define $\theta_{\max}(n) := \max\{\theta(G) : |V(G)| = n\}$ and $\theta'_{\max}(m) := \max\{\theta(G) : |E(G)| = m\}$

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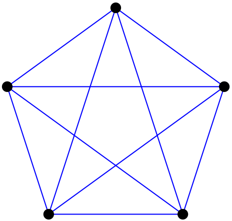
Question: What values does $\theta_{\max}(n)$ take?

Interval thickness example

$\theta(K_5) = 2$



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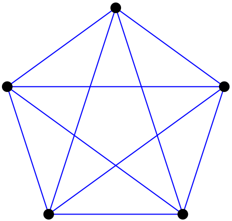

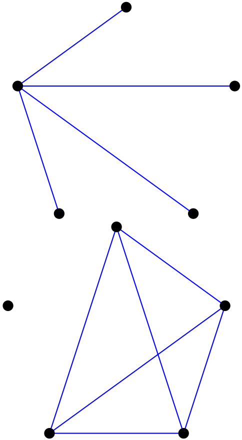

Previous bounds for $\theta_{\max}(n)$

- Asratian, Casselgren, and Petrosyan[1] proved $\theta_{\max}(n) \leq 2\lceil n/5 \rceil$

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The lower bound is less well studied. Asratian, Casselgren, and Petrosyan[1] also asked if for every $k \in \mathbb{Z}$ there is some graph G such that $\theta(G) = k$. Until now, no graph G was known with $\theta(G) \geq 3$.

[1] Armen S Asratian, Carl Johan Casselgren, and Petros A Petrosyan. "Decomposing graphs into interval colorable subgraphs and no-wait multi-stage schedules". In: *Discrete Applied Mathematics* (2022).

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Our results

We prove the following result.

Theorem

There is a universal constant c such that for every n,

$$c\frac{\log n}{\log\log n} \leq \theta_{\max}(n) \leq n^{5/6+o(1)}$$

Overview of the upper bound

Theorem

$$\theta_{\max}(n) \leq n^{5/6+o(1)}$$
$$\theta'_{\max}(m) \leq m^{5/11+o(1)}$$

Observation

Forests and bipartite regular graphs are interval-colourable.

Idea: a graph is either dense enough to contain a large bipartite regular graph, or sparse enough that a spanning tree contains *most* of its edges.

Overview of the upper bound

We use the 2 following results.[3][4]

Theorem (Rödl, Wysocka)

Let $\gamma : \mathbb{N} \to [0, 1/2)$ satisfy $\gamma(n) = \omega(n^{-1/3})$ as $n \to \infty$. Then every n-vertex graph with at least γn^2 edges contains an $\Omega(\gamma^3 n)$-regular subgraph.

Theorem (Dean, Hutchinson, Scheinerman)

The arboricity of any graph on m edges is at most $\sqrt{\frac{m}{2}}$.

[3] Vojtech Rödl and Beata Wysocka. "Note on regular subgraphs". In: *Journal of Graph Theory* 24.2 (1997), pp. 139–154.

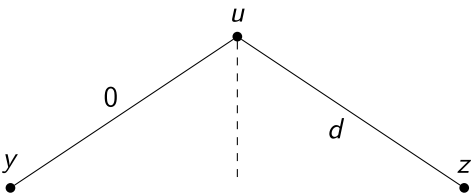
[4] Alice M Dean, Joan P Hutchinson, and Edward R Scheinerman. "On the thickness and arboricity of a graph". In: *Journal of Combinatorial Theory, Series B* 52.1 (1991), pp. 147–151.

Overview of the lower bound

We will make use of the following observation by Sevastianov.

Observation

Let G be an interval colourable graph and let $U \subseteq V(G)$. Suppose that there exists $d \in \mathbb{N}$ such that for all distinct $v, w \in U$ there is a path P in G from v to w such that $\sum_{x \in V(P)} d(x) \leq d$. Then for all $u \in V(G)$, we have $|N(u) \cap U| \leq d$.



Overview of the lower bound

Lemma

Fix $\alpha \in (0, 1/2]$ *and let* a *and* n *be integers satisfying* $n \geq \max\{1000(\log(a)+1)/\alpha, a+1\}$. *Then there is a bipartite graph* G *on parts* A *and* B *of sizes* a *and* n *respectively satisfying the following.*

1. *For all* $x \in A$, $d(x) = \lfloor \alpha n \rfloor$.
2. *For each* $\delta \in (0, 1]$ *with* $\delta \geq 10a^{-1/3}\alpha^{-1}$, *if* H *is a subgraph of* G *with at least* $\alpha\delta an$ *edges, then there exist* $A' \subseteq A$ *and* $B' \subseteq B$ *with* $|A'| \leq 1/\alpha$ *and* $|B'| \geq \delta n/16$ *such that the induced subgraph* $H[A' \cup B']$ *has diameter at most* 6.

Overview of the lower bound on $\theta_{\max}(n)$

- $|A_i| = \sqrt{n}$, $|B| = n$, $\alpha_i = 2^{-i}$, and construct $G_i = G[A_i \cup B]$ according to previous lemma.

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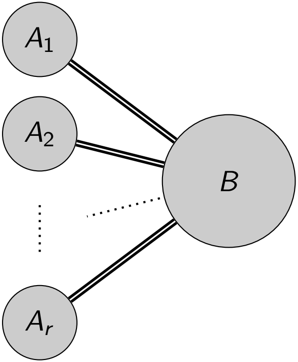

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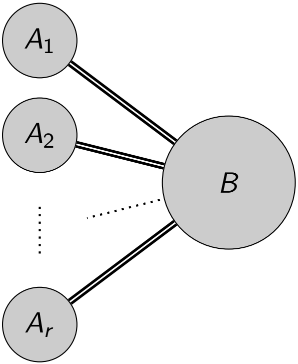

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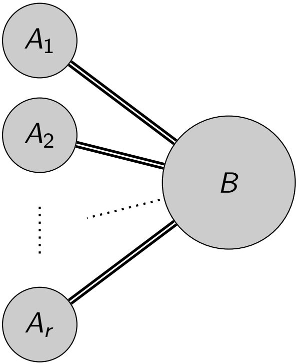
- Get contradiction by previous lemma and observation.



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Further research and open problems

There are many possible areas of further work here.

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- When is the Erdős-Renyi random graph $G(n, p)$ interval colourable? What about the random bipartite graph?
- For which a, b is it the case that all (a, b)-biregular graphs (i.e. $V = A \cup B$, degree of $x \in A$ is a, degree of $y \in B$ is b) are interval colourable?