Robust Algorithmics for NP-hard Problems on Weighted Graphs

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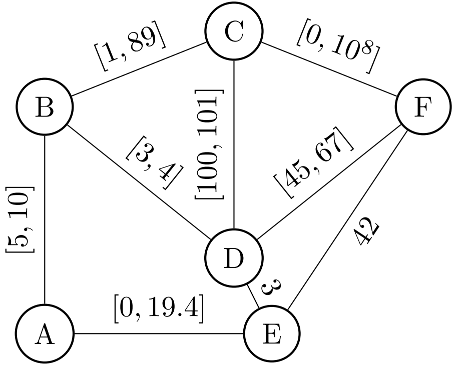
JGA 2023

Robust Travelling Salesman



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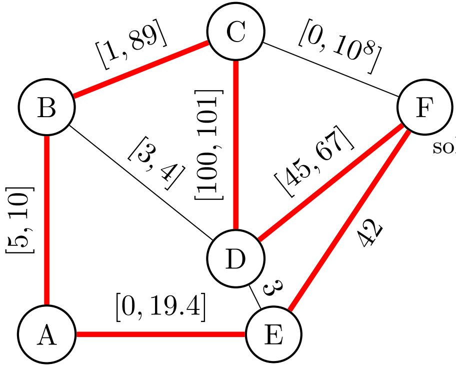

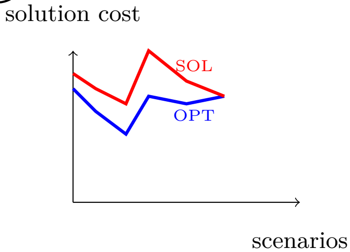

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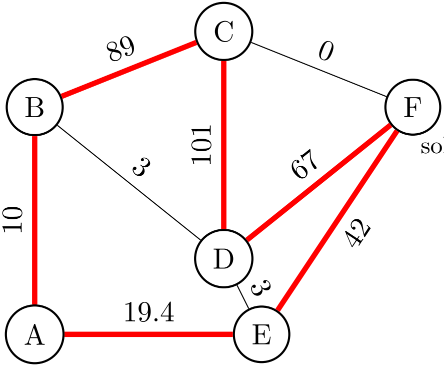

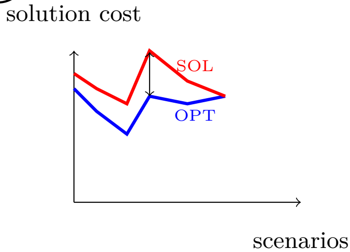

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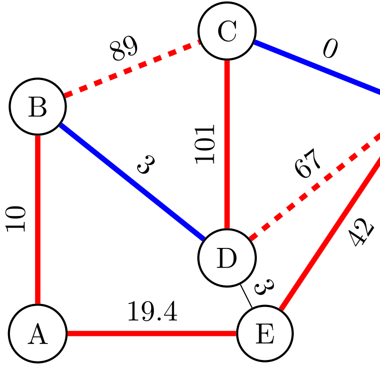

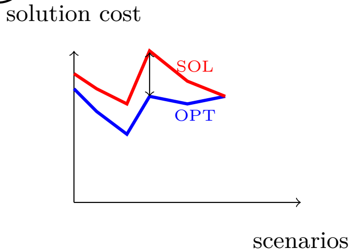

Regret: $SOL - OPT = 89 + 67 - (3 + 0) = 153.$

Robust approximation

We follow [GMP23].

- Minimizing regret is NP-hard.
- MR: minimal regret.
- Impossible to guarantee $\forall d, \mathit{regret}(d) \leq \beta MR$!
- (α, β)-robust approximation: $\forall d, \mathit{regret}(d) \leq \alpha \mathrm{opt}_d + \beta MR$.

$$
\begin{array}{lll}
\min & r & \\
\text{s.t.} & Ax \geq b & \text{(problem constraints)} \\
\forall d \in [\ell, u] & d^\top x \leq \mathrm{opt}_d + r & \text{(regret constraints)} \\
& x \geq 0 &
\end{array}
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Robust LP

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Robust LP

For polynomial problems

Theorem ([Kas08])

Any polynomial-time edge selection problem admits a $(0, 2)$-robust approximation algorithm:

$$\forall d, regret(d) \leq 2MR.$$

Take the optimal solution for $\frac{\ell+u}{2}$.

The central theorem

Theorem ([GMP23])

If an edge-selection problem:

- *has integrality gap* δ*;*
- *has approximation ratio* γ*;*
- *admits a* (α, β)*-approximate separation oracle for the robust LP;*

then it has a $(\alpha\delta\gamma, \beta\delta\gamma + \gamma)$*-approximate robust algorithm.*

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Approximate separation oracles

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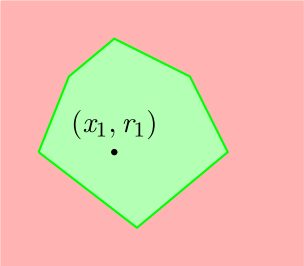


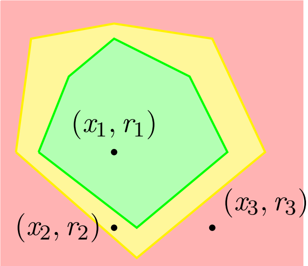
Definition (Approximate Separation Oracle)

A (α, β)-*approximate separation oracle* for the robust LP takes (x, r) and either

1. Guarantees that (x, r) is such that x is a fractional solution of the problem, and $\forall d$, $d^\top x \leq \alpha \text{opt}_d + \beta r$, or
2. Returns a d such that $d^\top x > \text{opt}_d + r$.

Approximate separation oracles

$$\begin{array}{lll} \min & & r \\ \text{s.t.} & & Ax \geq b \\ & \forall d \in [\ell, u] & d^\top x \leq \text{opt}_d + r \\ & & x \geq 0 \end{array}$$



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[GMP23]'s results

Definition (Steiner tree)

Given $S \subseteq V$, find a tree covering S of minimal weight.

- $(4.5, 3.75)$ for TSP (easy)
- $(2.78, 12.51)$ for Steiner Tree when $\ell = 0$ (somewhat easy)
- $(2755, 64)$ for Steiner Tree (difficult)

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The DOSS approach

Definition (DOSS)

We say that a problem $\mathcal{P}$ admits a α-*dense optimizable solution set* (DOSS) if there exists a polynomial-time algorithm that given instance $(G, [\ell, u])$, builds $\mathcal{S}$ and a polynomial-time function f such that:

1. For all $d \in [\ell, u]$, $\mathcal{S}$ contains an α-approximation to $\mathcal{P}$ on cost function d.
2. For all $d \in [\ell, u]$, $f(d) = \arg\min_{S \in \mathcal{S}} d(S)$.

Having an α-DOSS implies having a $(\alpha, 1)$-approximate separation oracle.

The DOSS approach for TSP

For TSP, a 2-DOSS is easy: $\mathcal{S} = \{2T \mid T \text{ a spanning tree}\}$.

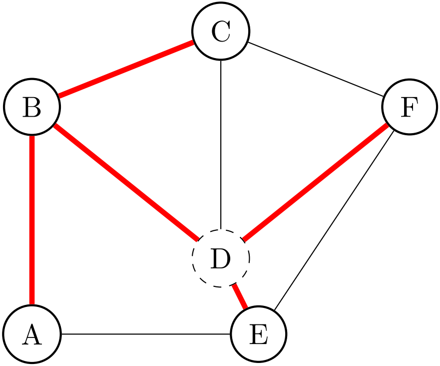
Can we find DOSSs for other problems?

"Most" problems do not have a DOSS :(

In TSP, one solution using a subset of the edges of another solution does not imply equality. In Steiner tree, it does.

"Most" problems do not have a DOSS :(

We show that in any tentative α-DOSS $\mathcal{S}$ for Steiner Tree, *every* Steiner tree is in $\mathcal{S}$. Thus the minimization operation is NP-hard.



$u > m\ell\alpha$

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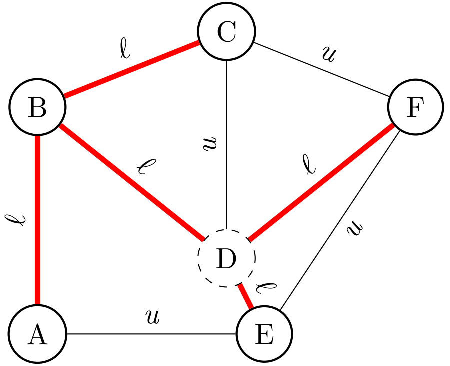

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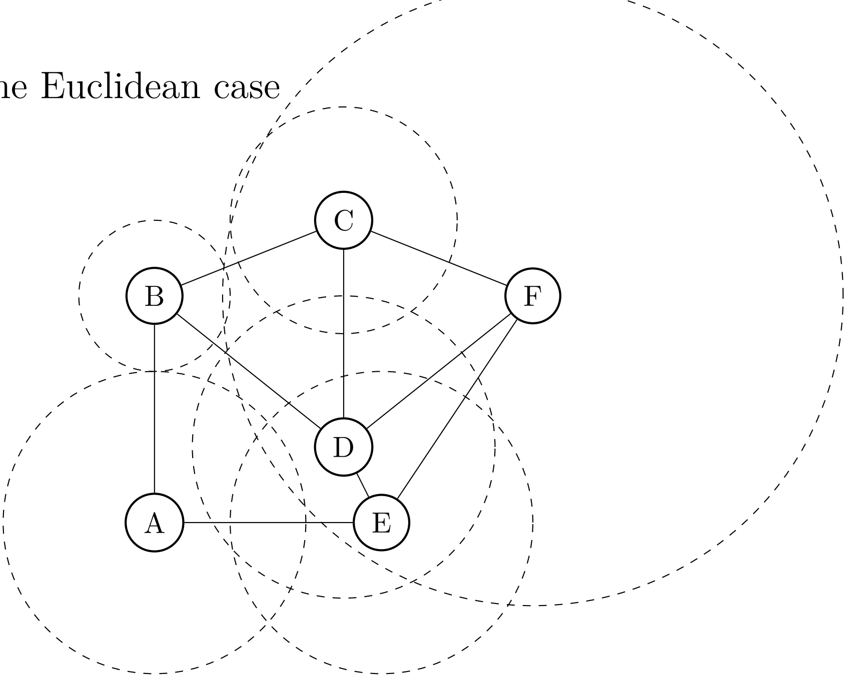
The Euclidean case



Conclusion

- Many interesting open questions!
- General framework for robustifying an approximation algorithm?
- Non-robust approximable problems with standard approximation?
- Euclidean cases, scheduling

Questions?

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1. [GMP23] Ganesh, Arun, Bruce M. Maggs, and Debmalya Panigrahi. “Robust Algorithms for TSP and Steiner Tree.” ACM Transactions on Algorithms 19, no. 2 (April 30, 2023): 1–37. https://doi.org/10.1145/3570957.
2. [Kas08] Kasperski, Adam. Discrete Optimization with Interval Data: Minmax Regret and Fuzzy Approach. Studies in Fuzziness and Soft Computing 228. Berlin: Springer, 2008.