Random embedding of bounded degree trees with optimal spread

Clément Legrand-Duchesne

LaBRI, Bordeaux

November 22, 2023

Joint work with Alp Müyesser, Paul Bastide

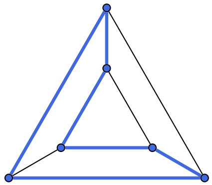
Hiking workshop

When does $H \subset G$?

- NP-complete (ex: Hamilton cycle)

When does $H \subset G$?

- NP-complete (ex: Hamilton cycle)
- What about sufficient conditions ?

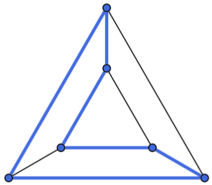


Dirac 1952

If $\delta(G) \geq n/2$ then G is hamiltonian

When does $H \subset G$?

- NP-complete (ex: Hamilton cycle)
- What about sufficient conditions ?



Dirac 1952

If $\delta(G) \geq n/2$ then G is hamiltonian

Dirac threshold of H

What is the infimum $\delta_{H,n}$ such that $\delta(G) \geq \delta_{H,n} \Rightarrow H \subset G$?

Embedding

Injection $\phi : H \to G$ such that $uv \in E(H) \Rightarrow \phi(u)\phi(v) \in E(G)$

Other questions

Embedding

Injection $\phi : H \to G$ such that $uv \in E(H) \Rightarrow \phi(u)\phi(v) \in E(G)$

Counting the embeddings

If $\delta(G) \geq \delta_{H,n}$, how many embeddings of H in G?

Other questions

Embedding

Injection $\phi : H \to G$ such that $uv \in E(H) \Rightarrow \phi(u)\phi(v) \in E(G)$

Counting the embeddings

If $\delta(G) \geq \delta_{H,n}$, how many embeddings of H in G?

Embedding in a random graph

For what p_n does $\mathbb{P}[H \subset G(n, p_n)] \geq \frac{1}{2}$?

Embedding

Injection $\phi : H \to G$ such that $uv \in E(H) \Rightarrow \phi(u)\phi(v) \in E(G)$

Counting the embeddings

If $\delta(G) \geq \delta_{H,n}$, how many embeddings of H in G?

Embedding in a random graph

For what p_n does $\mathbb{P}[H \subset G(n, p_n)] \geq \frac{1}{2}$?

Robustness: Embedding in a typical subgraph

$G * p$: Keep each edge of G with probability p
For what p'_n does $\mathbb{P}[H \subset G * p'_n] \geq \frac{1}{2}$ for all G with $\delta(G) \geq \delta_{H,n}$?

Unified approach

q-spread embedding

A distribution $\mathbb{P}$ over embedding $\phi : H \to G$ is q-spread if $\forall x_1, \dots x_s \in V(H)$, $\forall y_1, \dots y_s \in V(G)$,

$$\mathbb{P}[\forall i, \phi(x_i) = y_i] \leq q^s$$

Unified approach

q-spread embedding

A distribution $\mathbb{P}$ over embedding $\phi : H \to G$ is q-spread if $\forall x_1, \ldots x_s \in V(H)$, $\forall y_1, \ldots y_s \in V(G)$,

$$\mathbb{P}[\forall i, \phi(x_i) = y_i] \leq q^s$$

Typically, $q = \frac{C}{n}$

Unified approach

q-spread embedding

A distribution $\mathbb{P}$ over embedding $\phi : H \to G$ is q-spread if $\forall x_1, \dots x_s \in V(H)$, $\forall y_1, \dots y_s \in V(G)$,

$$\mathbb{P}[\forall i, \phi(x_i) = y_i] \leq q^s$$

Typically, $q = \frac{C}{n}$

Other point of view

Randomized algorithm embedding H progressively, with linearly many options at each step

Spreadness implies counting

If there is a q-spread distribution, then for all embedding ϕ_H,

$$\mathbb{P}[\phi = \phi_H] \leq q^{|H|}$$

Hence, # embedding $\geq q^{-|H|}$

Spreadness implies counting

If there is a q-spread distribution, then for all embedding ϕ_H,

$$\mathbb{P}[\phi = \phi_H] \leq q^{|H|}$$

Hence, $\#$ embedding $\geq q^{-|H|}$

Corollary

If there is a $\left(\frac{C}{n}\right)$-spread distribution, then G contains at least $\left(\frac{n}{C}\right)^n \geq n!/C^n$ copies of H

Spreadness implies counting

If there is a q-spread distribution, then for all embedding ϕ_H,

$$\mathbb{P}[\phi = \phi_H] \leq q^{|H|}$$

Hence, $\#$ embedding $\geq q^{-|H|}$

Corollary

If there is a $\left(\frac{C}{n}\right)$-spread distribution, then G contains at least $\left(\frac{n}{C}\right)^n \geq n!/C^n$ copies of H

Park, Pham 2022

Proved Kahn-Kalai conjecture

Corollary: spreadness implies robustness and random threshold

Embedding spanning trees of bounded degree

Komlos, Sarkozy, Szemeredi 1996

$\forall \Delta, \forall \alpha > 0$, for n large enough, $\delta(G) \geq (\frac{1}{2} + \alpha)n \Rightarrow G$ contains all n-vertex trees of maximum degree Δ

Embedding spanning trees of bounded degree

Komlos, Sarkozy, Szemeredi 1996

$\forall \Delta, \forall \alpha > 0$, for n large enough, $\delta(G) \geq (\frac{1}{2} + \alpha)n \Rightarrow G$ contains all n-vertex trees of maximum degree Δ

Pham, Sah, Sawhnhey, Simkin 23

$O(\frac{1}{n})$-spread distribution for perfect matchings, K_r-factor and spanning trees of bounded degree

Embedding spanning trees of bounded degree

Komlos, Sarkozy, Szemeredi 1996

$\forall \Delta, \forall \alpha > 0$, for n large enough, $\delta(G) \geq (\frac{1}{2} + \alpha)n \Rightarrow G$ contains all n-vertex trees of maximum degree Δ

Pham, Sah, Sawhnhey, Simkin 23

$O(\frac{1}{n})$-spread distribution for perfect matchings, K_r-factor and spanning trees of bounded degree

Bastide, L.-D., Müyesser 23+

$O(\frac{1}{n})$-spread distribution for spanning trees of bounded degree

- Avoids the Regularity Lemma
- Shorter and more flexible proof
- Better constants
- Generalizes painlessly to hypergraphs and digraphs

Sketch of proof

To reduce to nicer settings:

- Subdivide T in subtrees of controlled size
- Partition G randomly

Sketch of proof

To reduce to nicer settings:

- Subdivide T in subtrees of controlled size
- Partition G randomly

Intermediate steps

Embed T of max degree Δ in G with:

Final goal $|T| = |G| = n$ and $\delta(G) \geq (\frac{1}{2} + \alpha)n$

Sketch of proof

To reduce to nicer settings:

- Subdivide T in subtrees of controlled size
- Partition G randomly

Intermediate steps

Embed T of max degree Δ in G with:

Smaller trees $|T| = (1-\varepsilon)|G|$ and $\delta(G) \geq (\frac{1}{2} + \alpha)n$

Final goal $|T| = |G| = n$ and $\delta(G) \geq (\frac{1}{2} + \alpha)n$

Sketch of proof

To reduce to nicer settings:

- Subdivide T in subtrees of controlled size
- Partition G randomly

Intermediate steps

Embed T of max degree Δ in G with:

Boosted degree $|T| = (1-\varepsilon)|G|$ and $\delta(G) \geq (1-\rho)n$

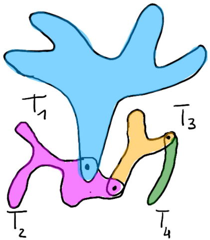
Smaller trees $|T| = (1-\varepsilon)|G|$ and $\delta(G) \geq (\frac{1}{2}+\alpha)n$

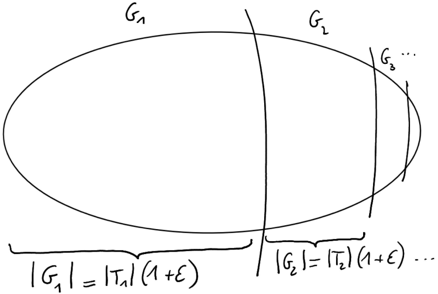
Final goal $|T| = |G| = n$ and $\delta(G) \geq (\frac{1}{2}+\alpha)n$

Final Goal

$|T| = n$ and $\delta(G) = (\frac{1}{2} + \alpha)n$

Assuming you know how to embed T in G with $|T| = (1 - \varepsilon)n$ and $\delta(G) \geq (\frac{1}{2} + \alpha)n$



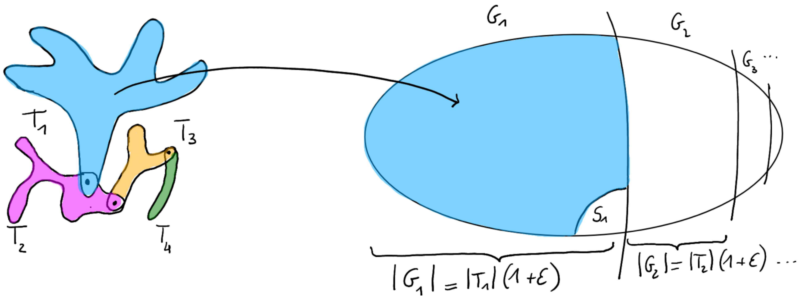


$|T_i| \approx$ constant fraction of $|T \setminus (\bigcup_{j<i} T_i)|$

Final Goal

$|T| = n$ and $\delta(G) = (\frac{1}{2} + \alpha)n$

Assuming you know how to embed T in G with $|T| = (1-\varepsilon)n$ and $\delta(G) \geq (\frac{1}{2} + \alpha)n$

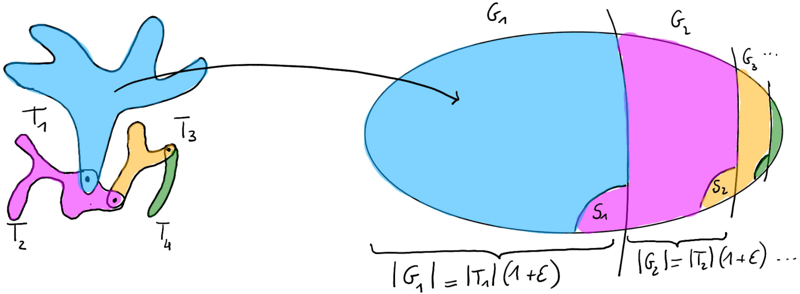


$|T_i| \approx$ constant fraction of $|T \setminus (\bigcup_{j<i} T_i)|$

Final Goal

$|T| = n$ and $\delta(G) = (\frac{1}{2} + \alpha)n$

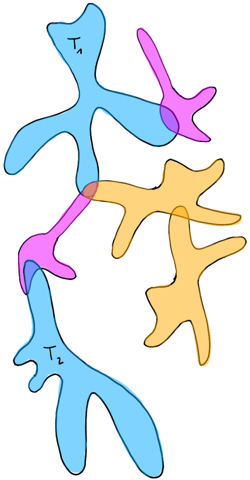
Assuming you know how to embed T in G with $|T| = (1-\varepsilon)n$ and $\delta(G) \geq (\frac{1}{2} + \alpha)n$



$|T_i| \approx$ constant fraction of $|T \setminus (\bigcup_{j<i} T_i)|$

Intermediate goal: Embed smaller trees

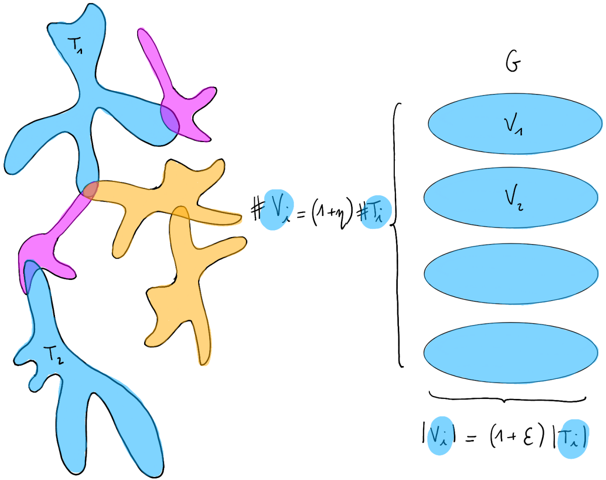
$|T| = (1 - \varepsilon)n$ and $\delta(G) = (\frac{1}{2} + \alpha)n$



- Same color = roughly the same size
- Linearly many T_i's in each color class

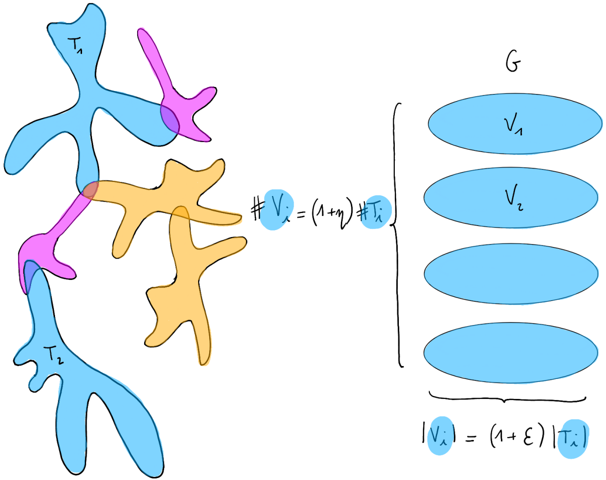
Intermediate goal: Embed smaller trees

$|T| = (1-\varepsilon)n$ and $\delta(G) = (\frac{1}{2}+\alpha)n$



- Same color = roughly the same size
- Linearly many T_i's in each color class

$|T| = (1 - \varepsilon)n$ and $\delta(G) = (\frac{1}{2} + \alpha)n$

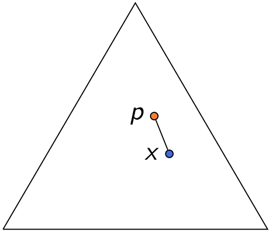


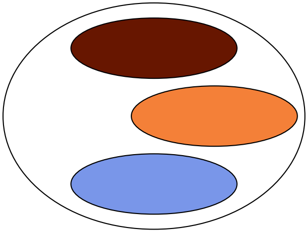
- Same color = roughly the same size
- Linearly many T_i's in each color class
- Almost all V_i are $\alpha/2$-Dirac
- Almost all (V_i, V_j) share good minimum degree

First step: Boosted minimal degree of G

$|T| = (1-\varepsilon)n$ and $\delta(G) = (1-\rho)n$ and respect the coloring

- For every color c, $|T \cap c| \leq (1-\eta)|V \cap c|$



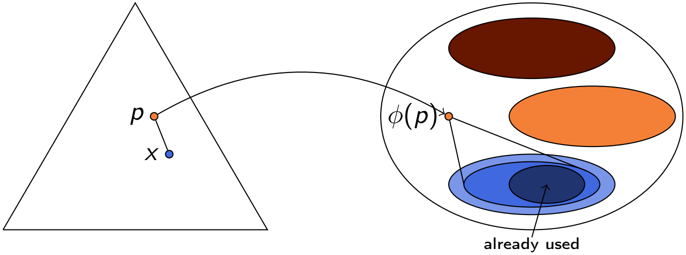


- $|N(\phi(p)) \cap c| \geq (1-\rho)|V(G) \cap c|$ with $\rho \ll \eta$
- linearly many choices for $\phi(x)$

First step: Boosted minimal degree of G

$|T| = (1-\varepsilon)n$ and $\delta(G) = (1-\rho)n$ and respect the coloring

- For every color c, $|T \cap c| \leq (1-\eta)|V \cap c|$
- Fix some BFS ordering of $V(T)$
- Following the ordering, embed each node x in its color class, among the free vertices of $N(\phi(p))$



- $|N(\phi(p)) \cap c| \geq (1-\rho)|V(G) \cap c|$ with $\rho \ll \eta$
- linearly many choices for $\phi(x)$

Future work

- Spread distribution for spanning grids when $\delta(G) \geq (\frac{1}{2} + \alpha)n$
  Subdivision arguments do not work as nicely
- Extend our result to graphs of bandwidth $o(n)$ when $\delta(G) \geq (\frac{1}{2} + \alpha)n$
  Probabilistic analysis more complex

Future work

- Spread distribution for spanning grids when $\delta(G) \geq (\frac{1}{2} + \alpha)n$
  Subdivision arguments do not work as nicely
- Extend our result to graphs of bandwidth $o(n)$ when $\delta(G) \geq (\frac{1}{2} + \alpha)n$
  Probabilistic analysis more complex

Thanks!