Directed hypergraph connectivity augmentation by hyperarc reorientation

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Edge connectivity

A graph $G = (V, E)$ is \textit{k-edge-connected} if and only if for all non-empty vertex set $X \neq V : d(X) \geq k$. 

This graph is 2-edge-connected.
Edge connectivity

A graph $G = (V, E)$ is $k$-edge-connected if and only if for all non-empty vertex set $X \neq V : d(X) \geq k$.

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Arc connectivity

A graph orientation $\vec{G} = (V, A)$ is $k$-arc-connected if and only if for all non-empty vertex set $X \neq V$: $d^-(X) \geq k$. 

Diagram of a graph orientation with arrows indicating directed edges.
Arc connectivity

A graph orientation $\vec{G} = (V, A)$ is $k$-arc-connected if and only if for all non-empty vertex set $X \neq V$: $d^-(X) \geq k$.

This orientation is 0-arc-connected.
Weak Orientation Theorem (Nash-Williams, 1960)

An undirected graph admits a $k$-arc-connected orientation if and only if it is $2k$-edge-connected.
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An undirected graph admits a $k$-arc-connected orientation if and only if it is $2k$-edge-connected.

Arc-Connectivity Augmentation (Ito et al., 2021)

Let $G = (V, E)$ be an undirected $(2k + 2)$-edge-connected graph, $D$ be a $k$-arc-connected orientation of $G$. Then, there exist orientations $D_1, D_2, \ldots, D_\ell$ of $G$ such that

- $D_i$ is obtained from $D_{i-1}$ by reversing an arc of $D_{i-1}$,
- $\ell \leq |V|^3$,
- $\lambda(D) \leq \lambda(D_1) \leq \lambda(D_2) \leq \ldots \leq \lambda(D_\ell) = k + 1$.

Furthermore, such orientations can be found in polynomial time.
The key idea of Ito et al.

Reversing an \((s, t)\)-path only changes the connectivity of vertex sets separating \(s\) and \(t\).
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We will iteratively reverse \((s, t)\)-paths connecting a minimal set \(S\) of in-degree \(k\) (\textit{in-tight} \(T^-\)) to a minimal set \(T\) of out-degree \(k\) (\textit{out-tight} \(T^+\)). We call \(s\) a \textit{source} and \(t\) a \textit{sink}. 
Connectivity loss by path-reversal.

Connectivity loss by arc-reversal.
Connectivity loss by path-reversal.

Connectivity loss by arc-reversal.

The dangers

Dangers:
- Connectivity loss by path-reversal.
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The dangers

Connectivity loss by path-reversal.

Connectivity loss by arc-reversal.

Useless paths.
How to preserve connectivity: path errors

We introduce a new family $R^-$ containing the minimum in-tight sets containing an out-tight set.
How to preserve connectivity: path errors

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Restraining our paths to $R$ prevents path-reversal connectivity loss. Thus, we search for $s$ and $t$ in $R$. 
How to preserve connectivity: arc errors

We reverse our \((s, t)\)-path from end to start.
For any vertex set \(X\) entered that doesn’t contain \(t\), \(d^+(X)\) is temporarily decreased by 1.

\[\cdots \rightarrow \circ \rightarrow \circ \rightarrow \cdots\]

Step 1

\[\cdots \rightarrow \circ \rightarrow \circ \rightarrow \cdots\]

Step 2

\[\cdots \leftarrow \circ \leftarrow \circ \leftarrow \cdots\]

Step 3
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\[
\begin{array}{ccc}
\cdots & \circlearrowright & \cdots \\
\text{Step 1} & & \\
\cdots & \circlearrowleft & \cdots \\
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\cdots & \circlearrowleft & \cdots \\
\text{Step 3} & & \\
\end{array}
\]

Our \((s, t)\)-path must not enter any out-tight set that doesn't contain \(t\).
How to do something: safe sources

A vertex $s$ is a safe source for $S \in \mathcal{M}^-$ if:

- (Safe) If $s \in Y \in \mathcal{T}^+$ then $S \subset Y$.
- (Useful) If $s \in Z$ such that $d^+(Z) = k + 1$ and $S \not\subseteq Z$ then there exists an out-tight set in $Z$ that doesn’t contain $s$.

![Diagram showing safe and not safe vertices](image)
Algorithm

- Pick a set \( R \in \mathcal{R}^- \) (If none, flip orientation).
- Pick a safe source \( s \) in a minimal set \( S \in \mathcal{T}^- \) with \( S \subseteq R \).
- Search for a minimum out-tight set \( T \) in \( R \).
  If the search enters an out-tight set, don’t exit it.
- Once the search gets inside a minimum out-tight set \( T \), find a safe sink \( t \) in \( T \).
- Reverse the search \((s, t)\)-path!

Because of the search rule, the path never leaves any out-tight set.

Repeat until no tight sets remain \( \implies \lambda(D) = k + 1 \).
Context: $G$ is 4-edge-connected and $\vec{G}$ is 1-arc-connected.
Let’s reconfigure!

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Hypergraph

A hypergraph $\mathcal{H} = (V, E)$ is composed of:

- Vertices in $V$
- Hyperedges in $E$, linking vertices together
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Partition-connectivity

$\mathcal{H}$ is $(k, k)$-partition-connected if for any partition $\mathcal{P}$ of $V$, at least $k|\mathcal{P}|$ hyperedges intersect at least 2 members of $\mathcal{P}$:

$$e_{\mathcal{H}}(\mathcal{P}) \geq k|\mathcal{P}|.$$

Partition-connectivity is a stronger version of edge-connectivity.
Directed Hypergraph

A directed hypergraph $\vec{H} = (V, A)$ is composed of:

- Vertices in $V$
- Hyperarcs in $A$ with a unique head vertex
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The hyperarc-connectivity $\vec{H}$ is $k$-hyperarc-connected if for any non-empty vertex set $X \neq V$, at least $k$ hyperarcs enter $X$. 
Towards generalization

Theorem on hypergraph orientations (Frank, Király, Király, 2003)

A hypergraph $\mathcal{H}$ admits a $k$-hyperarc-connected orientation if and only if it is $(k, k)$-partition-connected.
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Hyperarc-Connectivity Augmentation

Let $\mathcal{H} = (V, E)$ be a $(k + 1, k + 1)$-partition-connected hypergraph and $\mathcal{D}$ be a $k$-hyperarc-connected orientation of $\mathcal{H}$. Then, there exist orientations $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_\ell$ of $\mathcal{H}$ such that

1. $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by reorienting a hyperarc of $\mathcal{D}_{i-1}$,
2. $\ell \leq |V|^3$,
3. $\lambda(\mathcal{D}) \leq \lambda(\mathcal{D}_1) \leq \lambda(\mathcal{D}_2) \leq \ldots \leq \lambda(\mathcal{D}_\ell) = k + 1$.

Furthermore, such orientations can be found in polynomial time.
Our result

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Furthermore, such orientations can be found in polynomial time.

This is the first algorithm to compute a $k$-hyperarc-connected orientation of a hypergraph.
Frank’s result: path and cycle reversing

Reconfiguration of two $k$-arc-connected orientations (1982)

Given two $k$-arc-connected orientations $D, D'$ of a $2k$-edge-connected graph $G$, there exist $k$-arc-connected orientations $D = D_1, D_2, \cdots, D_{\ell} = D'$ of $G$ such that $D_i$ is obtained from $D_{i-1}$ by reversing a path or a cycle.

Applying this theorem arc-by-arc may decrease the connectivity by one temporarily.
Ito et al.’s result on reconfiguration

Reconfiguration reachability of $k$-arc-connected orientations

Given two $k$-arc-connected orientations $D, D'$ of a $(2k + 2)$-edge-connected graph $G$, there exist $k$-arc-connected orientations $D = D_1, D_2, \cdots, D_\ell = D'$ of $G$ such that $D_i$ is obtained from $D_{i-1}$ by reversing an arc of $D_{i-1}$. Furthermore, such orientations can be found in polynomial time.
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We augment $D$ and $D'$ to $(k + 1)$-arc-connectivity, then we apply Frank’s reconfiguration algorithm arc-by-arc.
It works on hypergraphs

We can adapt the proof of Frank to work on hypergraph orientations, leading to the following generalization.

Reconfiguration reachability of $k$-hyper-connected orientations

Given two $k$-hyperarc-connected orientations $\mathcal{D}, \mathcal{D}'$ of a $(k + 1, k + 1)$-partition-connected hypergraph $\mathcal{H}$, there exist $k$-hyperarc-connected orientations $\mathcal{D} = \mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_\ell = \mathcal{D}'$ of $\mathcal{H}$ such that $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by reorienting an hyperarc of $\mathcal{D}_{i-1}$.

Furthermore, such orientations can be found in polynomial time.
We generalized the results of Ito et al. to hypergraphs:

- We provided the first combinatorial algorithm for computing a $k$-hyperarc-connected orientation of a hypergraph.
- We show it is possible to reconfigure a $k$-hyperarc-connected orientation of a hypergraph into any other, if the hypergraph is $(k + 1, k + 1)$-partition-connected.

Open questions:

- Our upper bound on the number of reorientated hyperarcs is $|V|^3$. Can we do lower? (maybe $|V|^2$)
- The target when augmenting is $d^-(X) \geq k$. For which $f$ can we replace $k$ with $f(X)$?
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Merci!