Directed hypergraph connectivity augmentation by hyperarc reorientation

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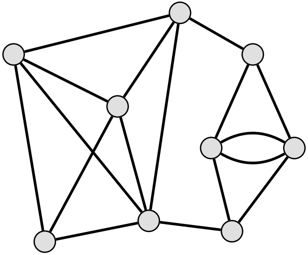
Laboratoire des Sciences pour la Conception, l'Optimisation et la production

Connectivity

Edge connectivity

A graph $G = (V, E)$ is k-**edge-connected** if and only if for all non-empty vertex set $X \neq V : d(X) \geq k$.



G-SCOP

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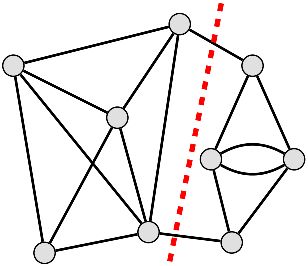
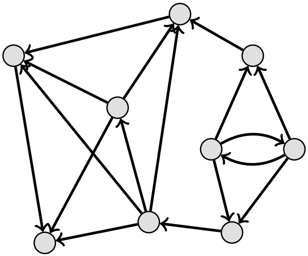


This graph is 2-edge-connected.

Connectivity

Arc connectivity

A graph orientation $\vec{G} = (V, A)$ is k-**arc-connected** if and only if for all non-empty vertex set $X \neq V$: $d^{-}(X) \geq k$.



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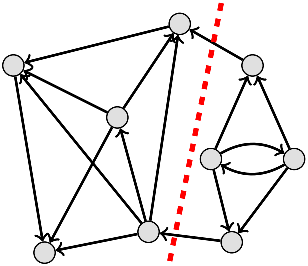

This orientation is 0-arc-connected.

Augmentation results

Weak Orientation Theorem (Nash-Williams, 1960)

An undirected graph admits a k-arc-connected orientation if and only if it is $2k$-edge-connected.

G-SCOP

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Arc-Connectivity Augmentation (Ito et al., 2021)

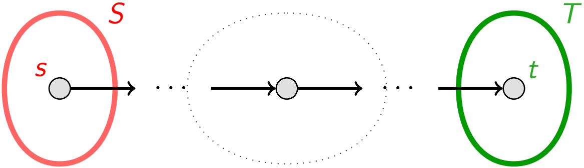
Let $G = (V, E)$ be an undirected $(2k+2)$-edge-connected graph, D be a k-arc-connected orientation of G.
Then, there exist orientations $D_1, D_2, \ldots, D_\ell$ of G such that

- D_i is obtained from D_{i-1} by reversing an arc of D_{i-1},
- $\ell \leq |V|^3$,
- $\lambda(D) \leq \lambda(D_1) \leq \lambda(D_2) \leq \ldots \leq \lambda(D_\ell) = k+1$.

Furthermore, such orientations can be found in polynomial time.

The key idea of Ito et al.

Reversing an (s, t)-path only changes the connectivity of vertex sets separating s and t.



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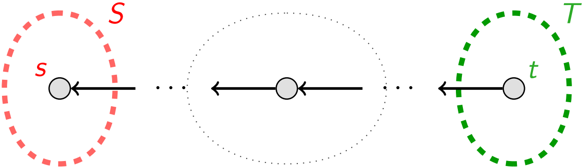
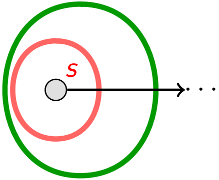
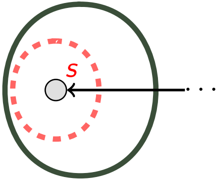


We will iteratively reverse (s, t)-paths connecting a minimal set S of in-degree k (**in-tight** $\mathcal{T}^-$) to a minimal set T of out-degree k (**out-tight** $\mathcal{T}^+$).
We call s a **source** and t a **sink**.

The dangers

Connectivity loss by path-reversal.





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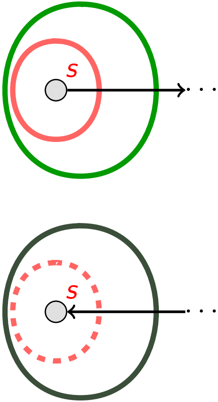
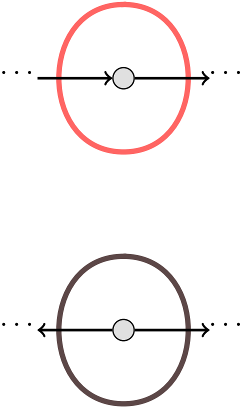


Connectivity loss by arc-reversal.

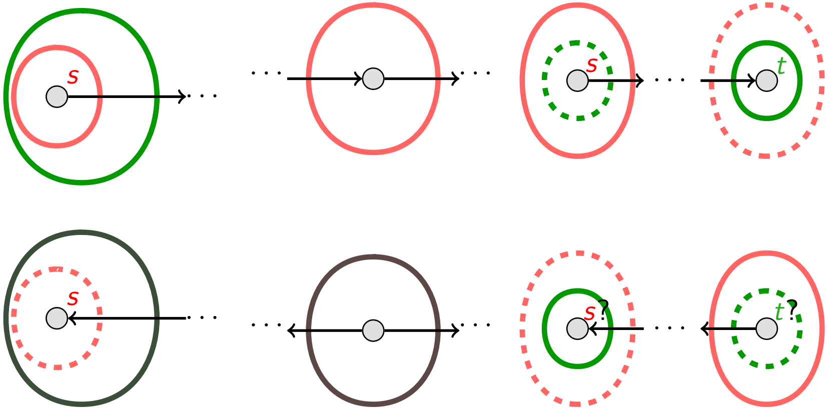


The dangers

Connectivity loss by path-reversal.

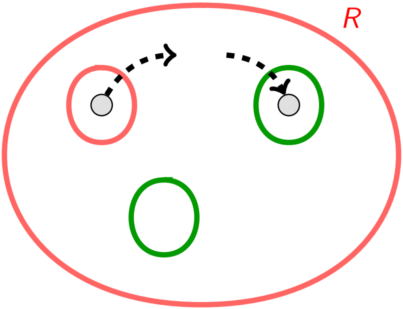
Connectivity loss by arc-reversal.

Useless paths.



How to preserve connectivity: path errors

We introduce a new family $\mathcal{R}^-$ containing the minimum in-tight sets containing an out-tight set.



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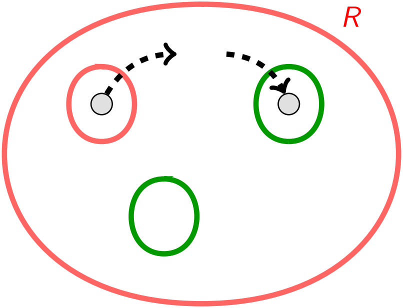
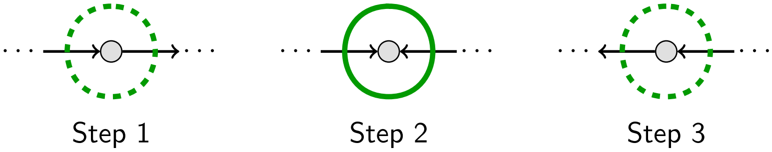


Restraining our paths to R prevents path-reversal connectivity loss.
Thus, we search for s and t in R.

How to preserve connectivity: arc errors

We reverse our (s, t)-path from end to start.
For any vertex set X entered that doesn't contain t, $d^+(X)$ is temporarily decreased by 1.



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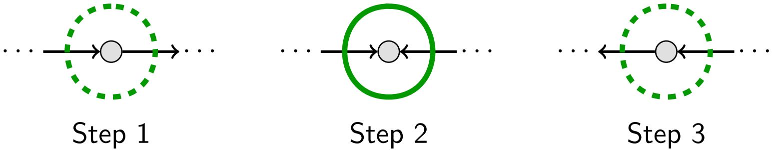
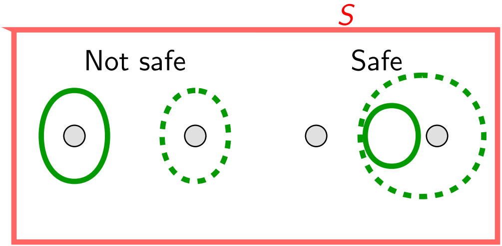


Our (s, t)-path must not enter any out-tight set that doesn't contain t.

How to do something: safe sources

A vertex s is a safe source for $S \in \mathcal{M}^-$ if:

- (Safe) If $s \in Y \in \mathcal{T}^+$ then $S \subset Y$.
- (Useful) If $s \in Z$ such that $d^+(Z) = k + 1$ and $S \not\subseteq Z$ then there exists an out-tight set in Z that doesn't contain s.



Algorithm

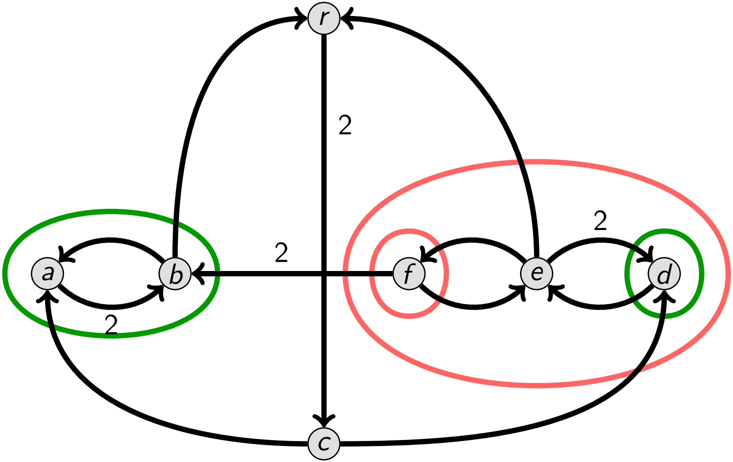
- Pick a set $R \in \mathcal{R}^-$ (*If none, flip orientation*).
- Pick a safe source s in a minimal set $S \in \mathcal{T}^-$ with $S \subseteq R$.
- Search for a minimum out-tight set T in R.
  If the search enters an out-tight set, don't exit it.
- Once the search gets inside a minimum out-tight set T, find a safe sink t in T.
- Reverse the search (s, t)-path!

Because of the search rule, the path never leaves any out-tight set.

Repeat until no tight sets remain $\implies \lambda(D) = k + 1$.

Let's reconfigure!

Context: G is 4-edge-connected and $\vec{G}$ is 1-arc-connected.



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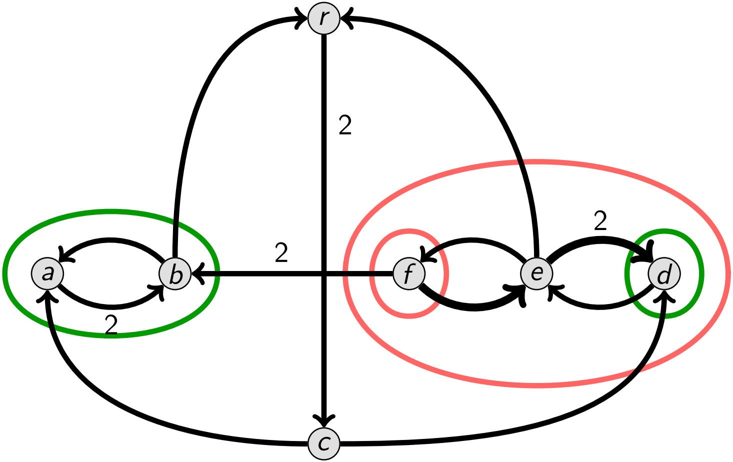

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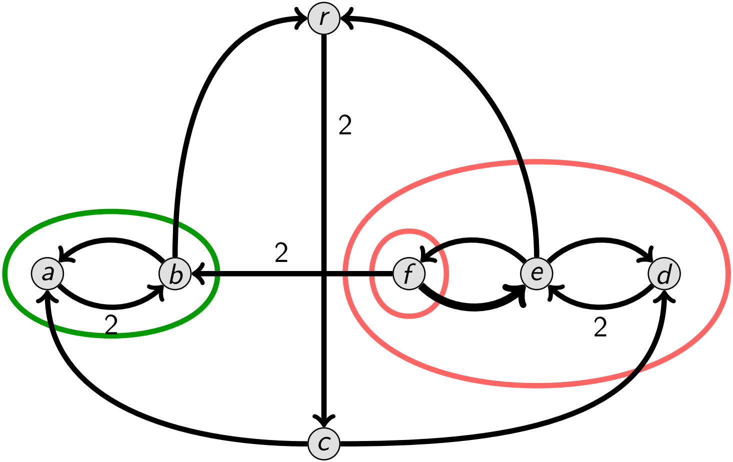

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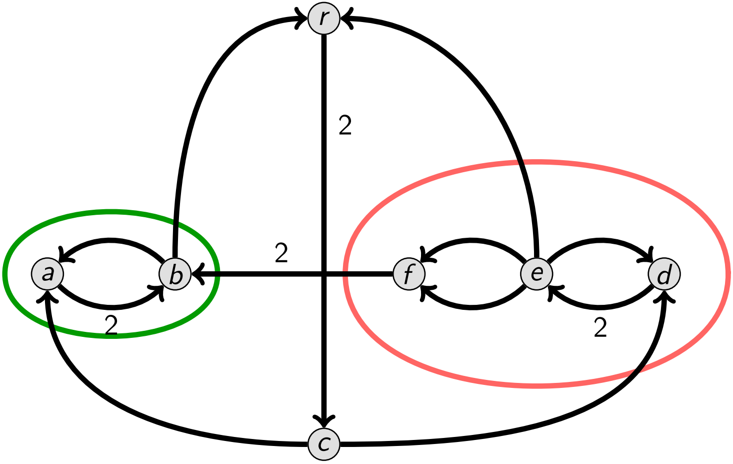

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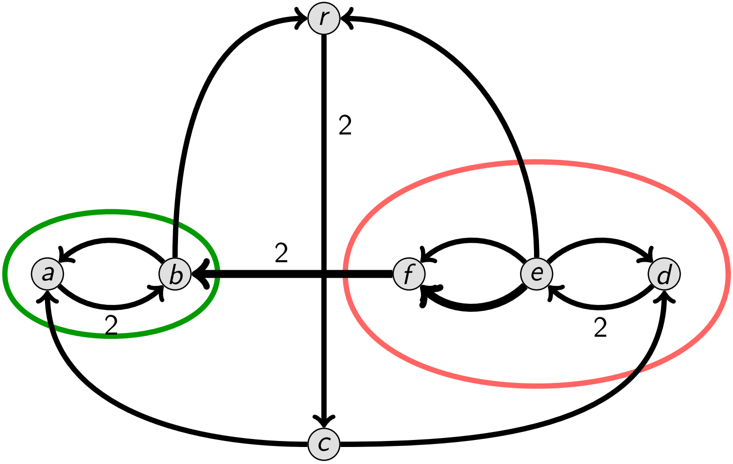

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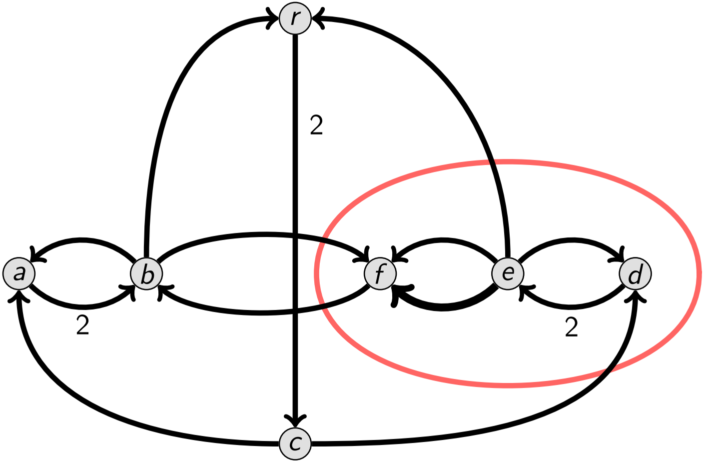

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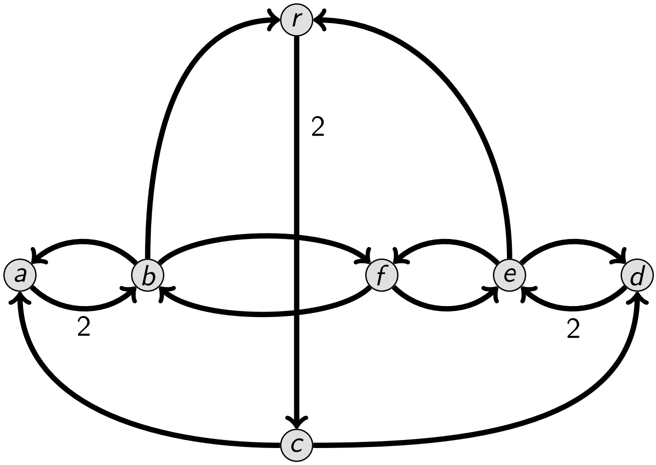
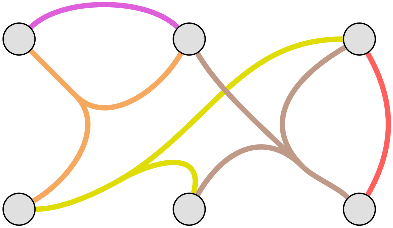


Hypergraphs

Hypergraph

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is composed of:

- Vertices in V
- Hyperedges in $\mathcal{E}$, linking vertices together



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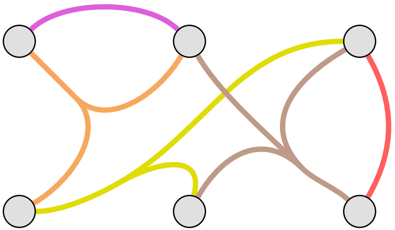


Partition-connectivity

$\mathcal{H}$ is (k,k)-**partition-connected** if for any partition $\mathcal{P}$ of V, at least $k|\mathcal{P}|$ hyperedges intersect at least 2 members of $\mathcal{P}$: $e_{\mathcal{H}}(\mathcal{P}) \geq k|\mathcal{P}|$.

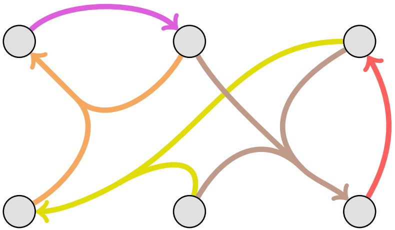
Partition-connectivity is a stronger version of edge-connectivity.

Directed Hypergraphs

Directed Hypergraph

A directed hypergraph $\vec{\mathcal{H}} = (V, \mathcal{A})$ is composed of:

- Vertices in V
- Hyperarcs in $\mathcal{A}$ with a unique head vertex



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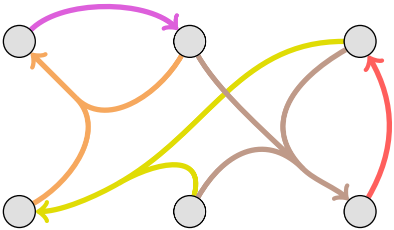

Hyperarc-connectivity

$\vec{\mathcal{H}}$ is k-**hyperarc-connected** if for any non-empty vertex set $X \neq V$, at least k hyperarcs enter X.

Towards generalization

Theorem on hypergraph orientations (Frank, Király, Király, 2003)

A hypergraph $\mathcal{H}$ admits a k-hyperarc-connected orientation if and only if it is (k,k)-partition-connected.

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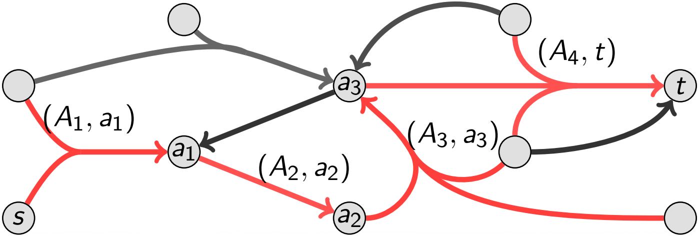
Most of the previous ideas work for connectivity augmentation!
Instead of finding good paths, we find good hyperpaths and reverse them.



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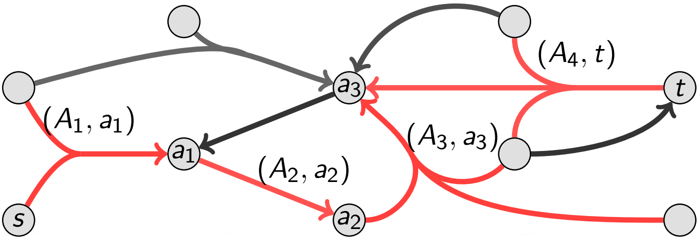

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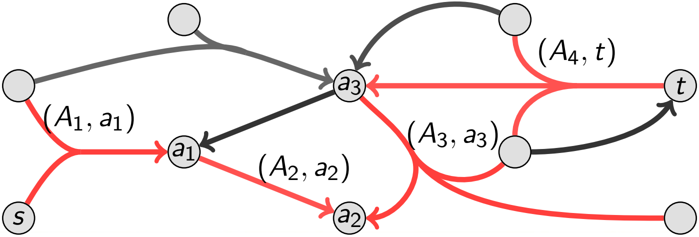

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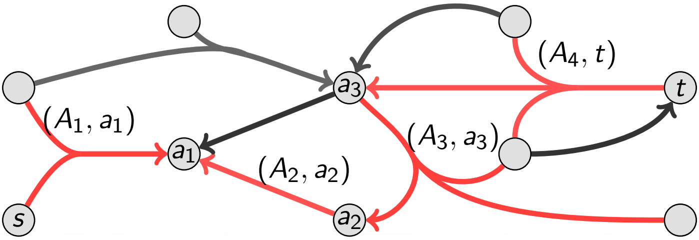

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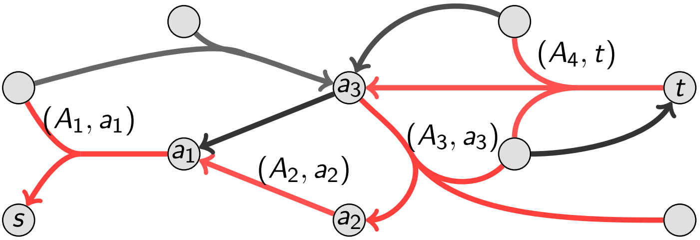

Our result

Hyperarc-Connectivity Augmentation

Let $\mathcal{H} = (V, E)$ be a $(k+1, k+1)$-partition-connected hypergraph and $\mathcal{D}$ be a k-hyperarc-connected orientation of $\mathcal{H}$.
Then, there exist orientations $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_\ell$ of $\mathcal{H}$ such that

- $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by reorienting a hyperarc of $\mathcal{D}_{i-1}$,
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Furthermore, such orientations can be found in polynomial time.

G-SCOP

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Furthermore, such orientations can be found in polynomial time.

This is the first algorithm to compute a k-hyperarc-connected orientation of a hypergraph.

G-SCOP

Frank's result : path and cycle reversing

Reconfiguration of two k-arc-connected orientations (1982)

Given two k-arc-connected orientations D, D' of a $2k$-edge-connected graph G, there exist k-arc-connected orientations $D = D_1, D_2, \cdots, D_\ell = D'$ of G such that D_i is obtained from D_{i-1} by reversing a path or a cycle.

Applying this theorem arc-by-arc may decrease the connectivity by one temporarily.

Ito et al.'s result on reconfiguration

Reconfiguration reachability of k-arc-connected orientations

Given two k-arc-connected orientations D, D' of a $(2k+2)$-edge-connected graph G, there exist k-arc-connected orientations $D = D_1, D_2, \cdots, D_\ell = D'$ of G such that D_i is obtained from D_{i-1} by reversing an arc of D_{i-1}.
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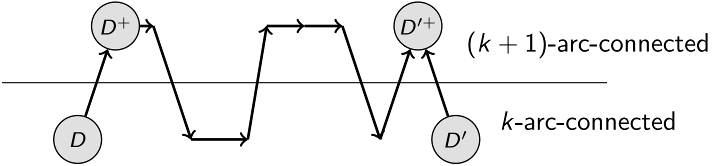
We augment D and D' to $(k+1)$-arc-connectivity, then we apply Frank's reconfiguration algorithm arc-by-arc.



It works on hypergraphs

We can adapt the proof of Frank to work on hypergraph orientations, leading to the following generalization.

Reconfiguration reachability of k-hyper-connected orientations

Given two k-hyperarc-connected orientations $\mathcal{D}, \mathcal{D}'$ of a $(k+1, k+1)$-partition-connected hypergraph $\mathcal{H}$, there exist k-hyperarc-connected orientations $\mathcal{D} = \mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_\ell = \mathcal{D}'$ of $\mathcal{H}$ such that $\mathcal{D}_i$ is obtained from $\mathcal{D}_{i-1}$ by reorienting an hyperarc of D_{i-1}.
Furthermore, such orientations can be found in polynomial time.

Conclusion

We generalized the results of Ito et al. to hypergraphs:

- We provided the first combinatorial algorithm for computing a k-hyperarc-connected orientation of a hypergraph.
- We show it is possible to reconfigure a k-hyperarc-connected orientation of a hypergraph into any other, if the hypergraph is $(k+1, k+1)$-partition-connected.

Open questions:

- Our upper bound on the number of reorientated hyperarcs is $|V|^3$. Can we do lower? (maybe $|V|^2$)
- The target when augmenting is $d^-(X) \geq k$. For which f can we replace k with $f(X)$?

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Merci !