On the parameterized complexity of non-hereditary relaxations of clique

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November 2023

Contents

Parameterized problem

NP-hard problem: No polynomial algo unless P=NP.

How to solve it efficiently in practice?

A parameterized problem is a couple (Π, λ) with:

- Π computational problem.
- λ is a **parameter**: ie $\lambda : \{\text{Instance of } \Pi\} \mapsto \mathbb{N}$

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The goal is to find exact algorithms to solve Π that are "fast" if the parameter is low.

Complexity classes

For all $C > 0$, the goal is to solve fast the problem:

$\underline{\Pi_{\lambda \leq C}}$:

Input: An instance x of Π $\underline{\text{with } \lambda(x) \leq C}$.

Ouput: The same question as Π on x.

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(Π, λ) is said to be:

- **XP** if $\forall C \geq 0, \exists d \geq 0, \Pi_{\lambda \leq C}$ is solvable in time $O(n^d)$.
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"Very easy"	P	$\Longrightarrow$	FPT	$\Longrightarrow$	XP	"Relatively easy"
	$\not\Updownarrow$		$\not\Updownarrow$		$\not\Updownarrow$	
"Relatively hard"	NP-hard	$\Longleftarrow$	W[1]-hard	$\Longleftarrow$	para-NP-hard	"Very hard"

$\not\Updownarrow$ (red): mutual exclusion unless P=NP.

$\not\Updownarrow$ (blue): mutual exclusion unless FPT=W[1].

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Definition of a clique:
S is a **clique** iff
$\forall u \in S, deg_S(u) \geq 1 \cdot (|S| - 1)$.

Definition of a γ-complete graph: $(\gamma \in]0,1[)$
S is a γ-**complete subgraph** iff
$\forall u \in S, deg_S(u) \geq \gamma \cdot (|S| - 1)$.

Problems studied

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s-CLUB:
Input: A graph G, an integer k.
Question: Does G have a s-club of size at least k?

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Classes of s-clubs and γ-complete graphs are not hereditary (stable by vertex deletion)! This raises a lot of technical issues.

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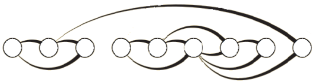

Figure: Every vertex has at most 2 neighbors on its left

***d*-elimination order:** if by removing the vertices from right to left, we always remove a vertex of degree $\leq d$.

State of the art

State of the art: *(Komusiewicz 2016.)*

Problem	k	ℓ	d
Clique	W[1]-h	FPT	FPT
2-club	FPT	FPT	para-NP-h (for $d = 6$)
s-club with $s \geq 3$	FPT	FPT	?
γ-complete-subgraph	W[1]-h $\forall \gamma \in [\frac{1}{2}, 1[$	?	?

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s-club is para-NP-hard

We prove that for all $s \geq 3$, the following problem is NP-hard:

s-**CLUB**$_{d \leq 3}$:
Input: A 3-degenerate graph G', an integer k'.
Question: Does there exists $S \subseteq V_{G'}$ with $|S| \geq k'$ and $\forall (u,v) \in S^2, \text{dist}_S(u,v) \leq s$?

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The *blue* vertices denote the original vertices (of G).

Vertices of other colors will be used to build G' starting from G.

Reduction from CLIQUE to s-CLUB$_{d \leq 3}$

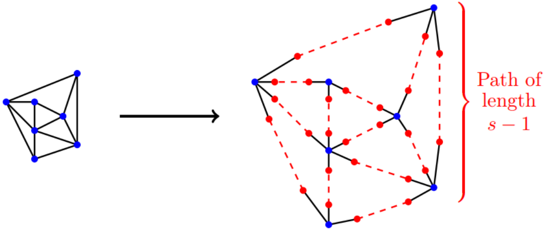

Figure: First step: lowering the degeneracy

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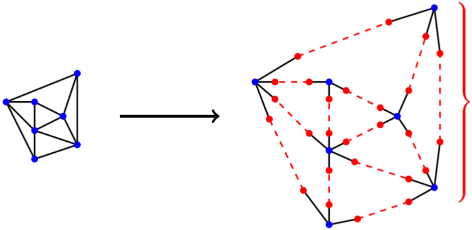

Figure: First step: lowering the degeneracy

- The graph obtained is 2-degenerate
- For blue vertices: Distance 1 in G $\iff$ Distance $s-1$ in G'

Yellow Vertex

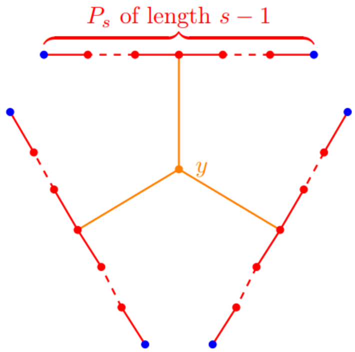

Figure: We add a vertex y linked to each "middle" red vertex

Works only for s odd (for the "middle" vertex to exist).

The reduction is correct

Let u and v two vertices of G':

- $dist_{G'}(u,v) \leq s$ if u and v are not both blue.
- $dist_{G'}(u,v) = s-1$ else if $\{u,v\} \in E_G$.
- $dist_{G'}(u,v) = s+1$ else if $\{u,v\} \notin E_G$: **Forbidden in a s-club.**

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S s-club in G' of size ... $\implies S \cap \{\text{Blue}\}$ is a clique in G of size k.

Degeneracy of G'

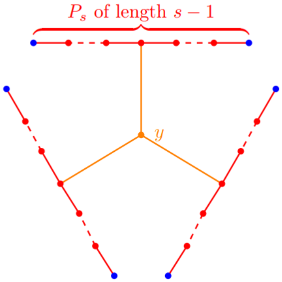

Figure: Graph G'

2-elimination order:

Blue and y ≤ Middle Red vertices ≤ Other red vertices

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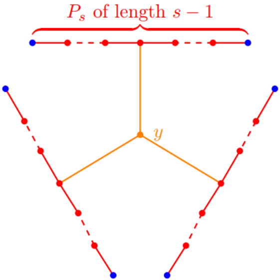

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Works only if $s \geq 5$. If $s = 3$, G' is only 3-degenerate.

s even

The ideas is the same but the reduction is more complicated.

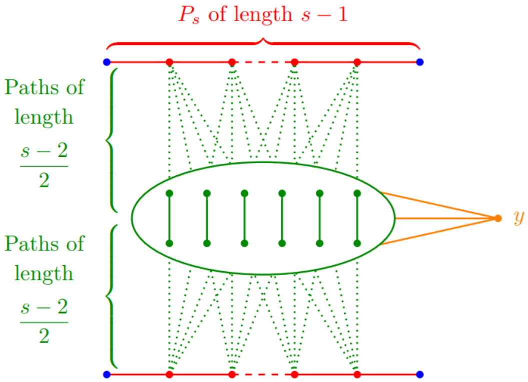


Figure: Reduction to s-club with s even

Contents

Parameterized reduction for d

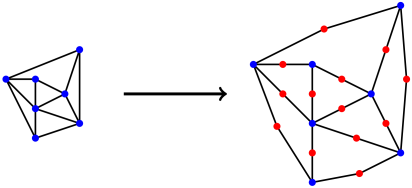


Figure: Widget to control degeneracy

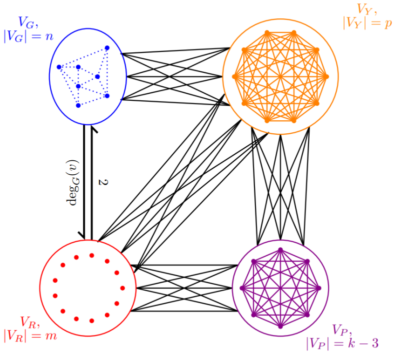
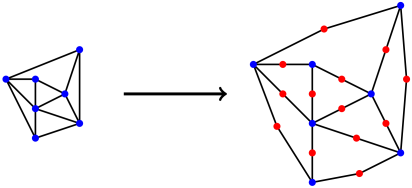


Figure: Whole reduction

This leads to (γ-COMPLETE-SUBGRAPH, d) being W[1]-hard.

Parameterized reduction for *d*



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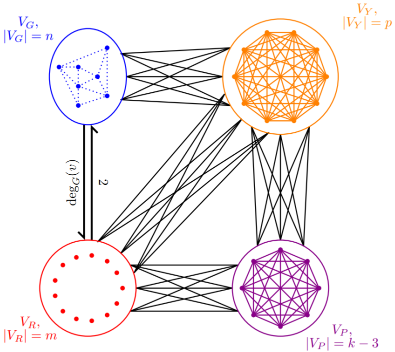

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We also get that (γ-COMPLETE-SUBGRAPH,k) is W[1]-hard.

Parameterized reduction for ℓ

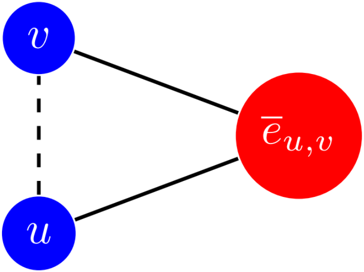
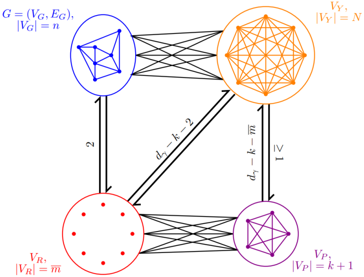


Figure: Red widget



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This leads to (γ-COMPLETE-SUBGRAPH,ℓ) being W[1]-hard.

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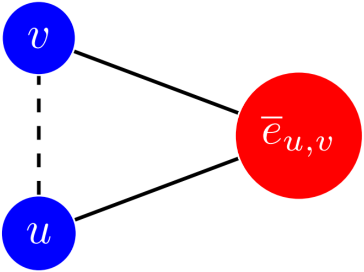

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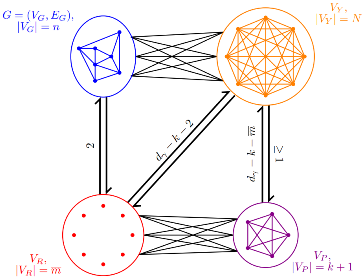

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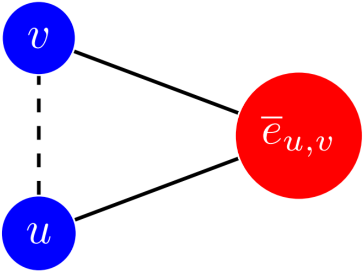
Surprising result! *s-PLEX:* each vertex has at most s non neighbors.

Parameterized reduction for ℓ



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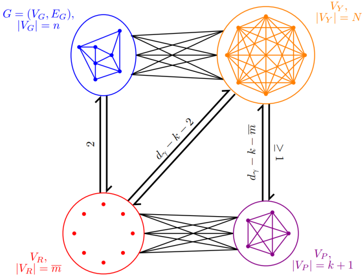

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(s-PLEX,ℓ) $\in$ FPT.

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Many other relaxations of CLIQUE.

My collaborators

Figure: J

Figure: G

Figure: A