Recognizing unit multiple interval graphs is hard

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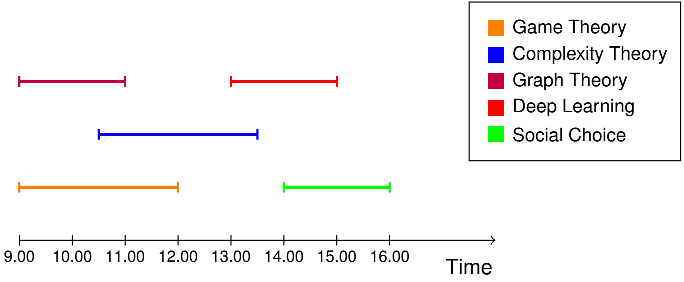
November 22, 2023

Motivating example

Suppose that we have a timetable of courses and we want to see which courses we can take:



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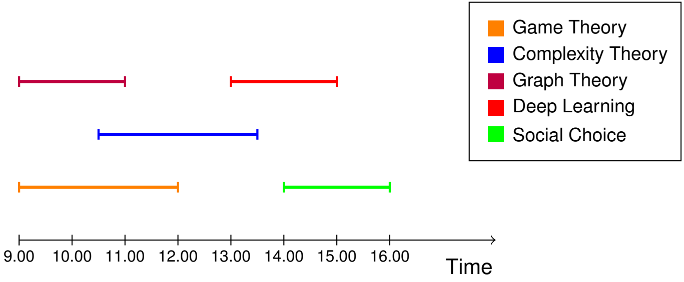
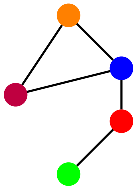


Graph representing the incompatibilities:



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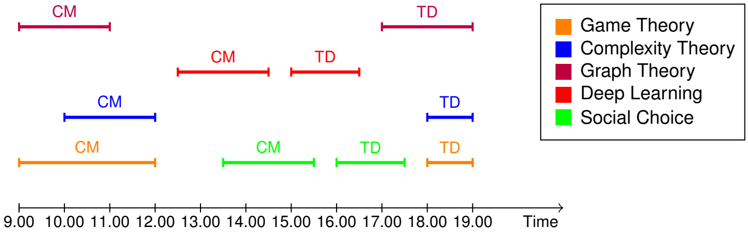
Suppose now that each course is split into a theoretical part and a practical part.



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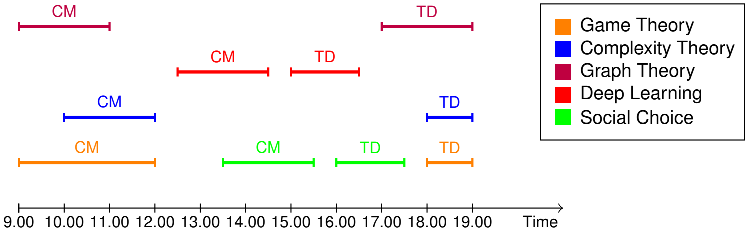

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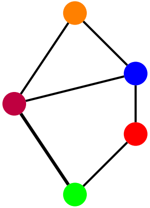

Motivating example

We can also ask for additional requirements, for example, for all the lectures to have the same duration.

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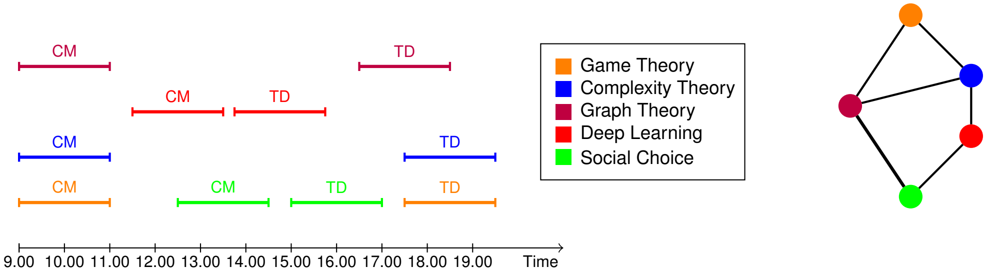

Problem Statement

Interval graphs

Definition

A graph is an **interval graph** if every vertex can be represented by an interval in such a way that there exists an edge between two vertices if and only if their corresponding intervals intersect.

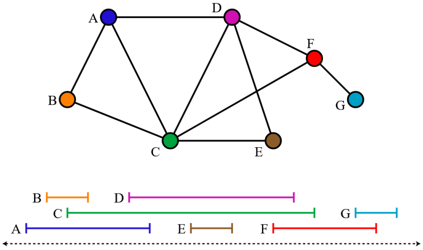

Figure 4: Interval graph and its associated interval representation.

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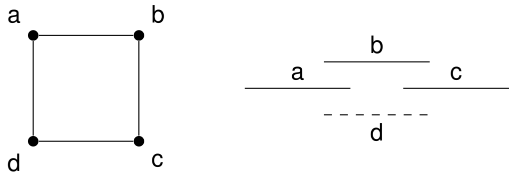


Figure 5: The cycle C_4 is not an interval graph.

Multiple interval graphs

Definition

A graph G is a d**-interval graph** if there exists a bijection from the vertices of G to a set of d-intervals (the union of d disjoint intervals) such that there exists an edge between two vertices if and only if their corresponding d-intervals intersect.

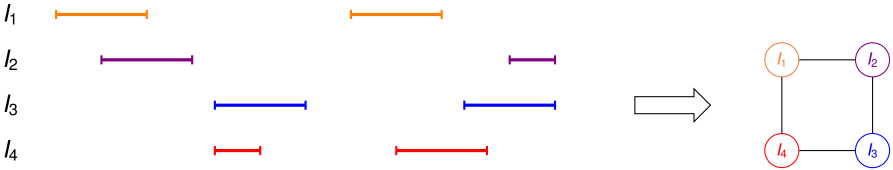


Figure 6: A 2-interval graph of four 2-intervals.

Unit interval graphs

Definition

- A (multiple) interval graph is **unit** if there exists a (multiple) interval representation where every interval has unit length.
- A (multiple) interval graph is **proper** if there exists a (multiple) interval representation where no interval properly contains another one.

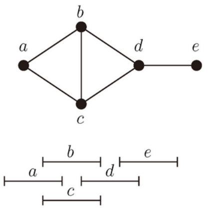


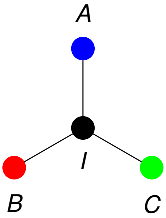
Figure 7: Unit/proper interval graph and its associated unit/proper interval representation.

Roberts' characterization

Theorem (Roberts' characterization of unit interval graphs)

For an undirected graph G, the following are equivalent:

1. *G is a proper interval graph.*
2. *G is a unit interval graph.*
3. *G is a $K_{1,3}$-free **interval** graph.*



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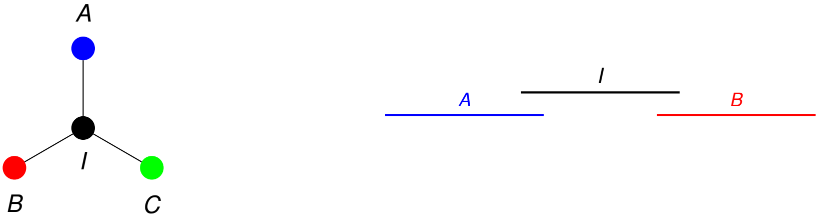

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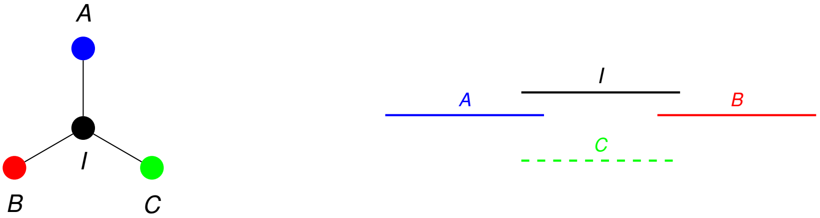

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Complexity of recognition of related classes

- Recognizing interval graphs and unit interval graphs can be done in linear time [Booth and Lueker, 76].
- Recognizing (unit [Jiang, 2013]) d-track interval graphs is NP-complete for any $d \geq 2$ [Gyárfás and West 95].
- Recognizing d-interval graphs is NP-complete for any $d \geq 2$ [West and Shmoys, 84].

	Interval graphs	Multiple track interval graphs	Multiple interval graphs
Unrestricted	Linear	NP-complete	NP-complete
Unit	Linear	NP-complete	?

Table 1: Known complexities of recognizing interval graphs and related classes.

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Table 1: Known complexities of recognizing interval graphs and related classes.

Complexity of recognizing unit d-interval graphs

Roadmap

1. We will first prove that a more general version of the problem, COLORED UNIT 2-INTERVAL RECOGNITION, is NP-hard.
2. Then, we will reduce COLORED UNIT 2-INTERVAL RECOGNITION to the recognition of unit 2-interval graphs.
3. Finally, we will obtain the hardness for unit *d*-interval graphs.

Colored unit 2-interval recognition

COLORED UNIT 2-INTERVAL RECOGNITION
Input: A graph $G = (V, E)$ and a coloring $\gamma : V \to \{\texttt{white}, \texttt{black}\}$.
Task: Decide whether G has an interval representation where:

- each white vertex is represented by a unit 2-interval,
- each black vertex is represented by a unit 1-interval.

We refer to this representation as a *colored unit 2-interval representation*.

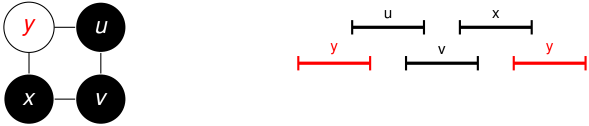

Figure 9: A C_4 with the given coloring is a colored unit 2-interval graph.

SAT $\leq_P$ COLORED UNIT 2-INTERVAL RECOGNITION

Complexity of recognizing colored unit 2-interval graphs

Theorem

COLORED UNIT 2-INTERVAL GRAPH RECOGNITION *is* NP*-complete.*

We reduce from SATISFIABILITY restricted to CNF-formulae such that:

- Every clause contains either 3 literals (3-clause) or 2 literals (2-clause).
- Each variable occurs exactly in three clauses, once positive in a 3-clause, once positive in a 2-clause and once negative in a 2-clause.

Example:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2) \wedge (x_3 \wedge \neg x_1) \wedge (x_2 \wedge \neg x_3)$$

Reduction

Let Φ be a formula. For every variable x_i, we introduce the following variable gadget:

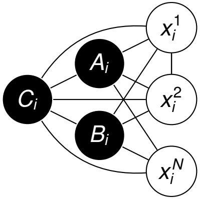

Figure 10: Variable gadget corresponding to a variable x_i.

Reduction

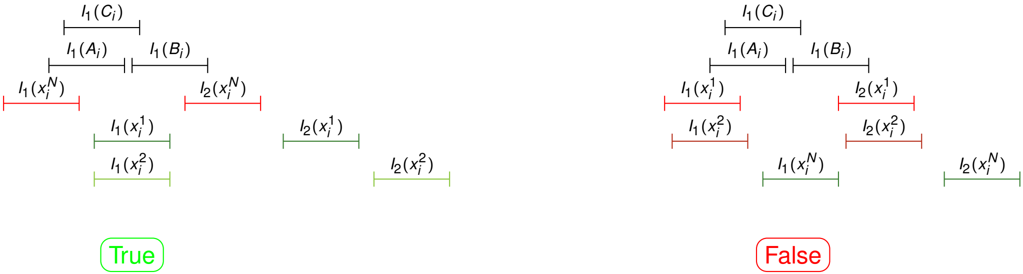

Figure 11: Representation of the variable gadget associated to the true value and the false value, respectively.

Reduction

For every 3-clause $C_\alpha = (x_i \vee x_j \vee x_k)$, we introduce the following clause gadget:

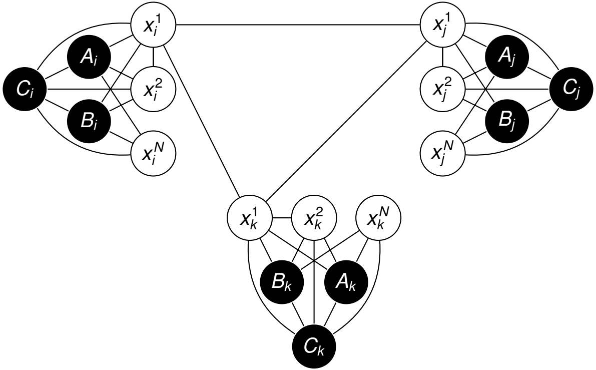

Figure 12: Clause gadget associated to a 3-clause $C_\alpha = (x_i \vee x_j \vee x_k)$.

Reduction

If Φ is satisfiable, then there exists a colored unit 2-interval representation of the constructed graph.

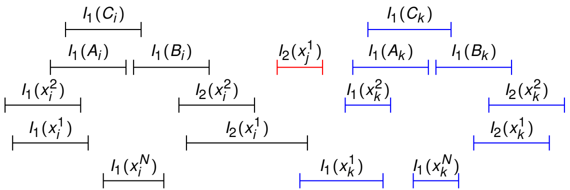

Figure 13: Representation of a 3-clause $(x_i \vee x_j \vee x_k)$, where x_i and x_k are set to false and x_j is set to true.

Φ is satisfiable if and only if there exists a colored unit 2-interval representation of the graph. Thus, COLORED UNIT 2-INTERVAL GRAPH RECOGNITION is NP-complete.

COLORED UNIT 2-INTERVAL GRAPH RECOGNITION $\leq_P$ UNIT 2-INTERVAL GRAPH RECOGNITION

Complexity of recognizing unit 2-interval graphs

Theorem

Recognizing unit 2-interval graphs is NP*-complete.*

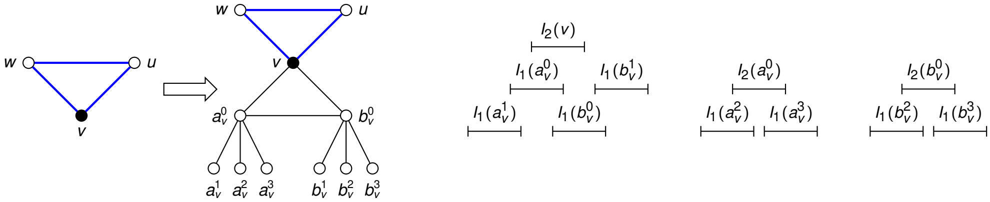

Figure 14: Gadget used to replace every black vertex v of G.

Complexity of recognizing unit d-interval graphs

Theorem

Recognizing depth r unit d-interval graphs is NP*-complete for every $r \geq 4$ and every $d \geq 2$.*

- Note that recognition problems are very different from optimization problems, and recognizing a class can be easier than recognizing a subclass of it.

Corollary

Unless the ETH fails, UNIT d-INTERVAL GRAPH RECOGNITION *does not admit an algorithm with running time $2^{o(|V|+|E|)}$.*

Open questions

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- We have obtained a lower bound for the running time of an algorithm for recognizing unit 2-interval graphs. However, the brute-force algorithm runs in $\mathcal{O}(2^{n^2})$. Is it possible to reduce this gap?
- We have shown that recognizing depth 4 unit d-interval graphs is NP-complete and it is known that the recognition of depth 2 unit d-interval graphs is polynomial-time solvable [Jiang, 2013], what happens for depth 3 unit d-interval graphs?

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Thank you