The structure of quasi-transitive graphs avoiding a minor with applications to the Domino Conjecture.

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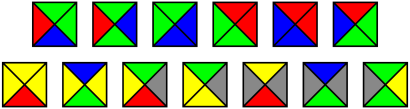
Wang tiling problem



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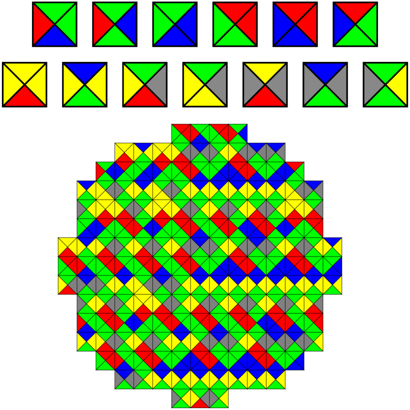

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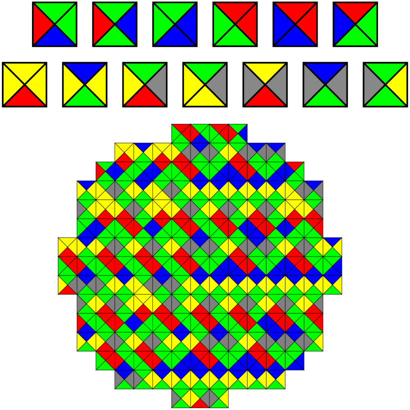

Theorem (Berger, '66)

The Wang tiling problem is undecidable.

Cayley graphs

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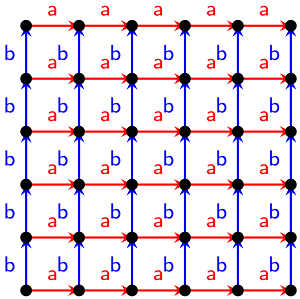
$\mathrm{Cay}(\mathbb{Z}^2, S)$,
with $S = \{(1,0), (-1,0), (0,1), (0,-1)\}$



Domino Problem on groups

Fix (Γ, S).
Pattern of $\mathrm{Cay}(\Gamma, S)$: coloring p of $\{1_\Gamma, s\}$ for some $s \in S$.
p appears in a vertex-coloring of $\mathrm{Cay}(\Gamma, S)$ if there is a pair $(w, w \cdot s)$ colored p.

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Domino problem on (Γ, S):

Input: a finite alphabet Σ and a finite set $\mathcal{F} = \{p_1, \dots, p_t\}$ of forbidden patterns.

Question: Is there a coloring $c : V(G) \to \Sigma$ avoiding $\mathcal{F}$?

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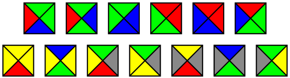

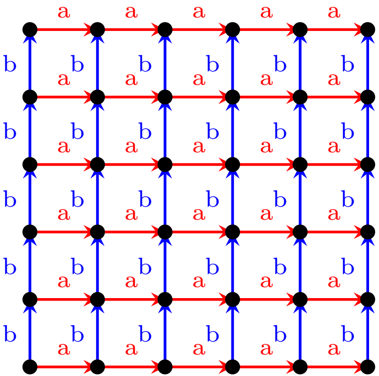

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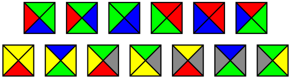
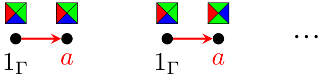
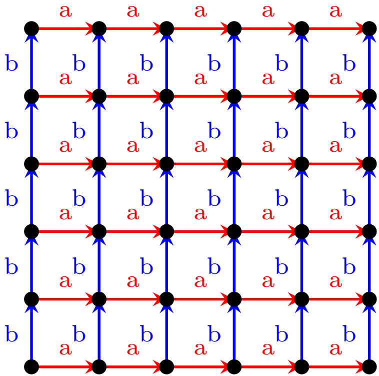


Forbidden patterns:





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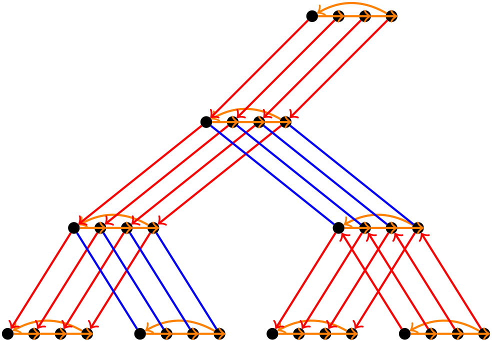

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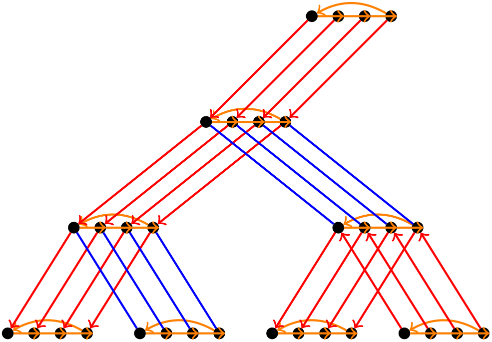

Conjecture (Ballier-Stein 2018)

The domino problem on Γ is decidable if and only if Γ is virtually-free.

Minor-excluded groups

A group is planar if one of its Cayley graphs is planar.
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Remark: G minor-excluded $\Leftrightarrow$ G is K_∞-minor free.

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Decidable on virtually-free groups;

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Theorem

The Domino conjecture is true for planar groups and more generally for minor-excluding groups.

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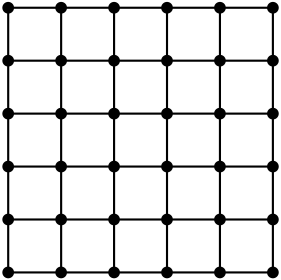

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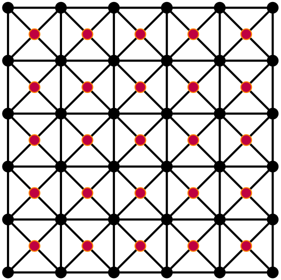

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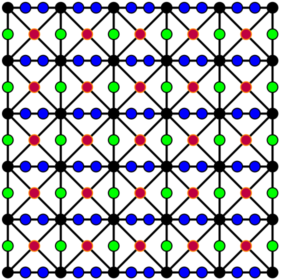
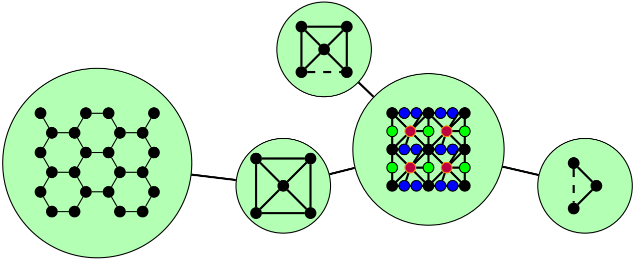


Main result

Theorem (finite/planar)

Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most k whose torsos are either finite or quasi-transitive 3-connected planar minors of G.



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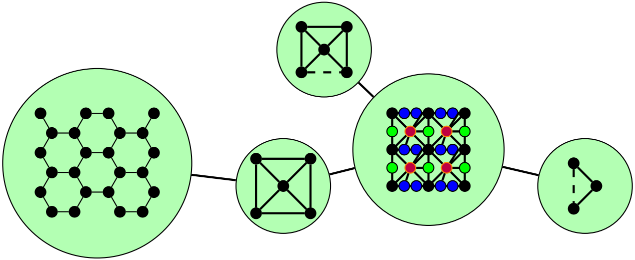

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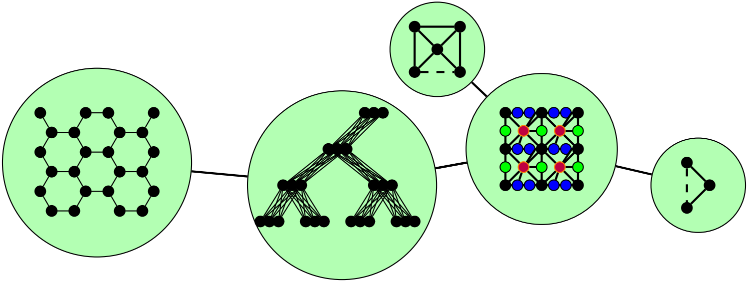
Let G be a quasi-transitive locally finite graph excluding K_∞ as a minor. Then there is an integer k such that G admits a canonical tree-decomposition $(T, \mathcal{V})$, of adhesion at most 3 whose torsos are quasi-transitive minors of G and have either treewidth at most k or are 3-connected planar.



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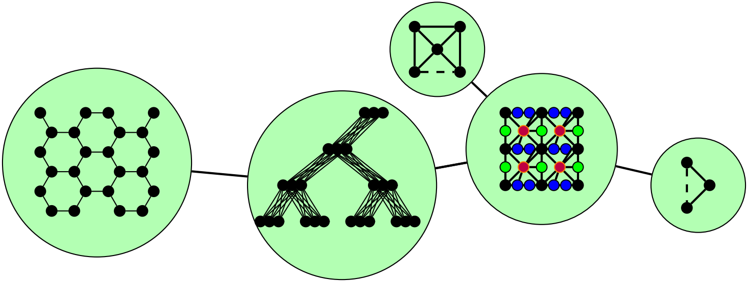

A graph application

Corollary

For every locally finite quasi-transitive graph G avoiding K_∞ as a minor, there is an integer k such that G is K_k-minor-free.

Generalizes [Thomassen '92] dealing with the 4-connected case.

Conclusion

- Prove results on groups by working in the more general world of quasi-transitive graphs.
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Questions:

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Merci!

Application: Finite presentability.

Theorem (Droms '06)

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Corollary

Every minor-excluding finitely generated group Γ *is finitely presented.*

Proof based on the approach of [Hamann '18]