Reconfiguration of Graph Homomorphisms (and Topology)

Joint works with (1) Moritz Mühlenthaler and Benjamin Lévêque, (2) Moritz Mühlenthaler and Mark H. Siggers

Thomas Suzan

Université Grenoble-Alpes, G-SCOP Laboratory, Grenoble, France

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Graph coloring and homomorphisms

A homomorphism $\alpha: G \rightarrow H$ is a map $V(G) \rightarrow V(H)$ such that $(uv) \in E(G) \Rightarrow (\alpha(u)\alpha(v)) \in E(H)$. $\alpha(u)$ is called the color of $u$. 

![Graph Diagram](image-url)
Graph coloring and homomorphisms

\[ k\text{-coloring of } G \iff \text{Graph homomorphism } \alpha: G \to K_k \]
Graph coloring and homomorphisms

$k$-coloring of $G \iff$ Graph homomorphism $\alpha : G \to K_k$

Definition

A homomorphism $\alpha : G \to H$ is a map $V(G) \to V(H)$ such that $(uv) \in E(G) \Rightarrow (\alpha(u)\alpha(v)) \in E(H)$. $\alpha(u)$ is called the color of $u$. 
For each arc $u \rightarrow v$ in $G$, $\alpha(u) \rightarrow \alpha(v)$ is an arc in $H$.

Homomorphism $G \rightarrow H = "H$-coloring" of $G$.

For $u \in V(G)$, $\alpha(u) = "color"$ of $u$
reflexive graph homomorphisms

Reflexive undirected graphs

Reflexive digraphs
**The $H$-recoloring problem**

**Definition**

**Instance:** Graph $G$ and two homomorphisms $\alpha, \beta : G \to H$.

**Question:** Can we change $\alpha$ to $\beta$ by recoloring the vertices one by one and keeping a homomorphism all along?
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Positive results (1)

$H$-Recoloring for loopless $H$ is polynomial if

- $H$ is $\{\square, \circlearrowright\}$-free [S., Mühlenenthaler, Lévêque 22]
  (generalizes $H$ is $\circlearrowright$-free [Wrochna 20], which
  generalizes $H = \triangle$ [Cereceda et al. 11])
Positive results (1)

*H*-Recoloring for loopless *H* is polynomial if

- *H* is \((\emptyset, \emptyset, \emptyset)\)-free [S., Mühlenenthaler, Lévêque 22] (generalizes *H* is \(\emptyset\)-free [Wrochna 20], which generalizes *H* = \(\emptyset\) [Cereceda et al. 11])
- *H* is a circular clique \(C_{p,q}\), where \(2 \leq p/q < 4\) [Brewster et al. 16]
Positive results (1)

\( H \)-Recoloring for loopless \( H \) is \textit{polynomial} if

- \( H \) is (\includegraphics{logo})-free [S., Mühlenthaler, Lévêque 22] (generalizes \( H \) is \includegraphics{logo}-free [Wrochna 20], which generalizes \( H = \includegraphics{logo} \) [Cereceda et al. 11])
- \( H \) is a circular clique \( C_{p,q} \), where \( 2 \leq p/q < 4 \) [Brewster et al. 16]
- \( H \) is a transitive tournament [Dochtermann & Singh 21]
H-Recoloring for reflexive $H$ is polynomial if

- $H$ is $(\circ, \boxtimes, \bigtriangleup)$-free [S., Mühlenenthaler, Lévêque 22]
  generalizes previous results:
- $H$ is a $(\circ, \boxtimes, \bigtriangleup)$-free reflexive digraph cycle [Brewster et al. 21]
- $H$ is reflexive undirected and of girth $\geq 5$ [Lee et al. 21]
Hardness results (3)

\(H\)-Recoloring is PSPACE-complete if

- \(H = K_n, n \geq 4\) [Bonsma & Cereceda 09]
- \(H = C_{p,q}\) for \(p/q \geq 4\) [Brewster et al. 16]
- \(H\) is a wheel \(W_k, k \geq 3, k \neq 4\) [Lee et al. 20]
- \(H\) is a \(K_{2,3}\)-free quadrangulation of the 2-sphere \(\neq \)
  [Lee et al. 20]
- \(H = \) [Wrochna 20]
The topological invariant

Main topological observation

*With the right assumptions on $H$, reconfiguration moves do not change the winding of cycles.*
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Main topological observation

*With the right assumptions on \( H \), reconfiguration moves do not change the winding of cycles.*

Main algorithmic idea for \( H \)-recoloring

The moves of one vertex determines the moves of all other vertices.
Example for $H$-Recoloring

$\beta(q) \xrightarrow{S_q} \alpha(q)$

$\beta(v) \xrightarrow{S_v} \alpha(v)$
Example for $H$-Recoloring

\[ \alpha(q) \xrightarrow{S_q} \beta(q) \]

\[ \alpha(v) \xrightarrow{S_v} \beta(v) \]

\[ \alpha \quad \rightarrow \quad \beta \]
Example for $H$-Recoloring

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Example for $H$-Recoloring

\[
\begin{align*}
\beta(q) & \rightarrow S_q & \alpha(q) \\
\beta(v) & \rightarrow S_v & \alpha(v)
\end{align*}
\]
The $H$-mixing problem

**Definition**

Col$(G, H)$

- **Vertices:** Homomorphisms $G \rightarrow H$
- **Edges:** single vertex recoloring.

**Instance:** A graph $G$.

**Question:** Is Col$(H)$ connected?
Remarks

$H$-Mixing is co-NP complete if:

- $H = K_3$. 


# Results (4)

$H$-Mixing:

- is polynomial when $H$ is a symmetric tree.
- can still be NP-hard if $H$ is an orientation of a tree.

$H$-Mixing is co-NP complete if:

- $H = K_3$.
- $H$ is a reflexive triangle-free, non-tree symmetric graph. (Kim, Lee and Siggers 22)
- $H$ is an irreflexive non tree square-free symmetric graph. (Siggers, Mühlenthaler, S. 23+). This is a particular case of a more general result for digraphs.
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Remark

The complexity of $K_k$-mixing is still unknown for $k \geq 4$. 
Topography for $H$-mixing [KLS 22]

Key idea

If $\alpha : G \rightarrow H$ and $\beta : G \rightarrow H$ wind a cycle of $G$ in different ways then $G$ is not $H$-mixing.
**Definition**

We say that $\alpha : G \rightarrow H$ is *flat* if $\alpha$ does not wind any cycle of $G$ around a cycle of $H$.

**Key idea**

If $\alpha : G \rightarrow H$ and $\beta : G \rightarrow H$ wind a cycle of $G$ in different ways then $G$ is not $H$-mixing.
Definition

We say that $\alpha: G \to H$ is flat if $\alpha$ does not wind any cycle of $G$ around a cycle of $H$.

Key idea

If $\alpha: G \to H$ and $\beta: G \to H$ wind a cycle of $G$ in different ways then $G$ is not $H$-mixing. In particular, if $\alpha: G \to H$ is flat and not $\beta$, then $G$ is not $H$-mixing.
Constructing a gadget [KLS 22]

Gadget $G^a$:

- There is a flat homomorphism $G^a \to H$.
- There is a non-flat homomorphism $G^a \to H$ iff $G \to K_3$. 

Adaptation of the proof

- In [KLS 22]: Underlying simplicial complex where triangles are flat (Clique complex).
- In [MSS 23+]: Realize the same topological structure as the in-neighborhood complex of an adapted gadget (and so encodes 3-coloring in the exact same way).
Last (big) steps

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Thank you!