Reconfiguration of Graph Homomorphisms (and Topology)

Joint works with (1) Moritz Mühlenthaler and Benjamin Lévêque, (2) Moritz Mühlenthaler and Mark H. Siggers

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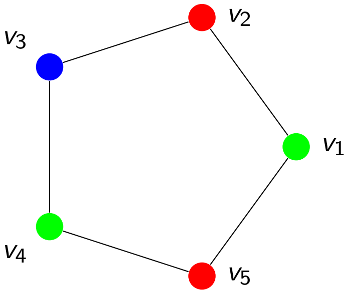
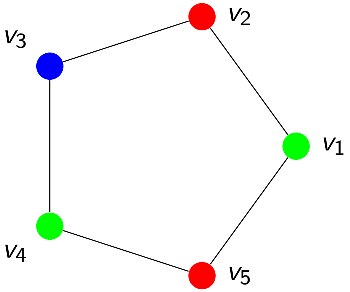
November 23, 2023

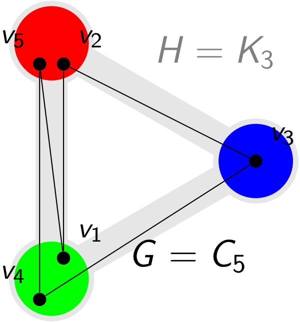
Graph coloring and homomorphisms



Graph coloring and homomorphisms

k-coloring of G $\iff$ Graph homomorphism $\alpha\colon G \to K_k$





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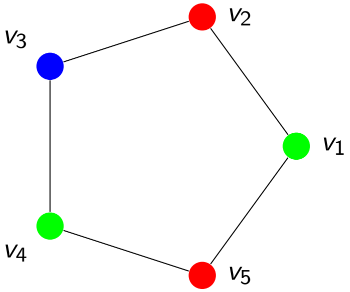
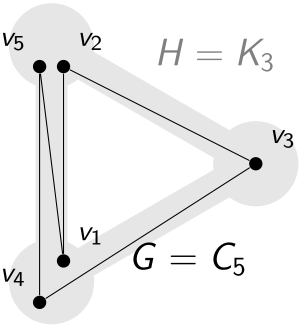
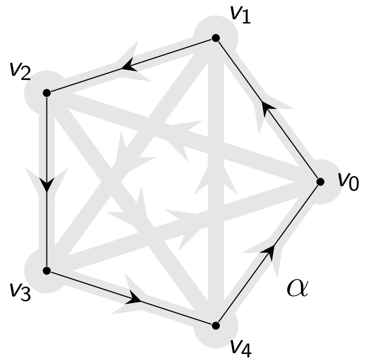




Definition

A homomorphism $\alpha\colon G \to H$ is a map $V(G) \to V(H)$ such that $(uv) \in E(G) \Rightarrow (\alpha(u)\alpha(v)) \in E(H)$. $\alpha(u)$ is called the *color* of u.

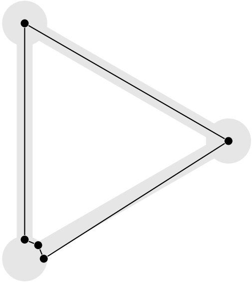
Digraph homomorphism



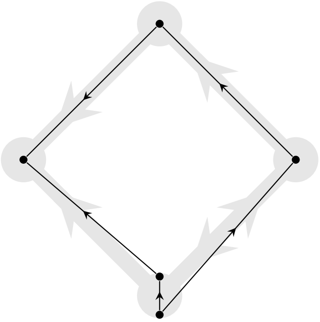
For each arc $u \to v$ in G, $\alpha(u) \to \alpha(v)$ is an arc in H.
Homomorphism $G \to H$ = "*H-coloring*" of G.
For $u \in V(G)$, $\alpha(u)$ = "*color*" of u

reflexive graph homomorphisms

Reflexive undirected graphs



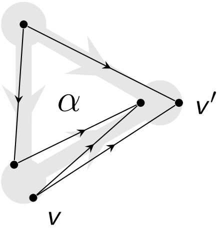
Reflexive digraphs

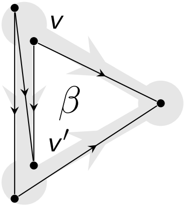


The H-recoloring problem

Definition

Instance: Graph G and two homomorphisms $\alpha, \beta \colon G \to H$.
Question: Can we change α to β by recoloring the vertices one by one and keeping a homomorphism all along?





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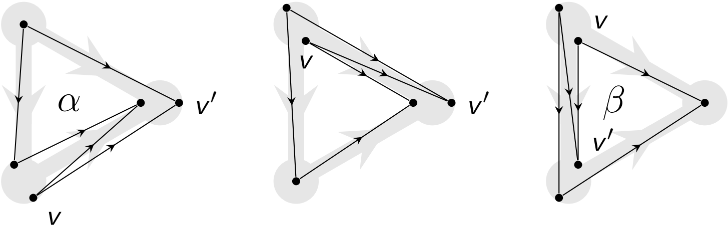
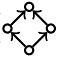
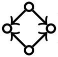
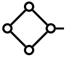
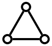


Positive results (1)

H-Recoloring for **loopless** H is **polynomial** if

- H is (, )-free [S., Mühlenthaler,Lévêque 22]

  (generalizes H is -free [Wrochna 20], which generalizes $H =$  [Cereceda et al. 11])

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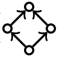
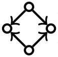
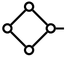
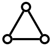
- H is (, )-free [S., Mühlenthaler,Lévêque 22]

  (generalizes H is -free [Wrochna 20], which

  generalizes $H =$  [Cereceda et al. 11])
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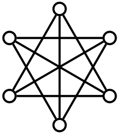

$C_{6,2}$

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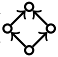
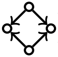
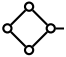
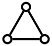
- H is (, )-free [S., Mühlenthaler,Lévêque 22]

  (generalizes H is -free [Wrochna 20], which

  generalizes $H =$  [Cereceda et al. 11])
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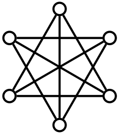
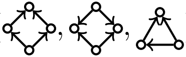
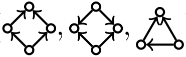


$C_{6,2}$

- H is a transitive tournament [Dochtermann & Singh 21]

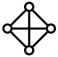
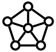
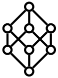
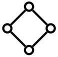
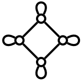
Positive results (2)

H-Recoloring for **reflexive** H is **polynomial** if

- H is ()-free [S., Mühlenthaler,Lévêque 22] generalizes previous results:
- H is a ()-free reflexive digraph cycle [Brewster et al. 21]
- H is reflexive undirected and of girth ≥ 5 [Lee et al. 21]

Hardness results (3)

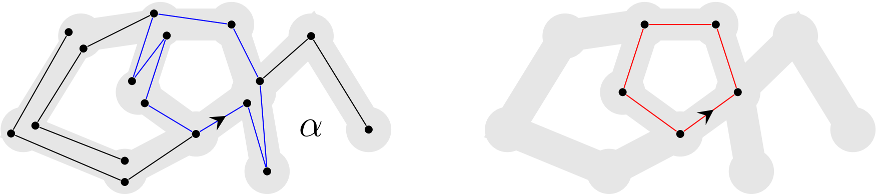
H-Recoloring is PSPACE-complete if

-  $H = K_n$, $n \geq 4$ [Bonsma & Cereceda 09]
- $H = C_{p,q}$ for $p/q \geq 4$ [Brewster et al. 16]
-  H is a wheel W_k, $k \geq 3$, $k \neq 4$ [Lee et al. 20]
-  H is a $K_{2,3}$-free quadrangulation of the 2-sphere $\neq$  [Lee et al. 20]
- $H =$  [Wrochna 20]

The topological invariant

Main topological observation

With the right assumptions on H, reconfiguration moves do not change the winding of cycles.



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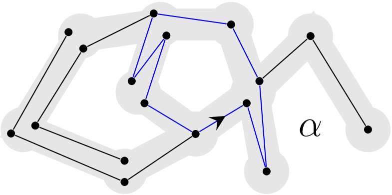
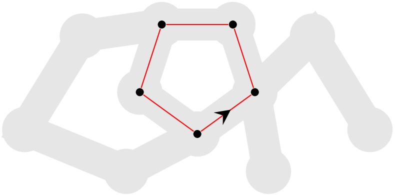
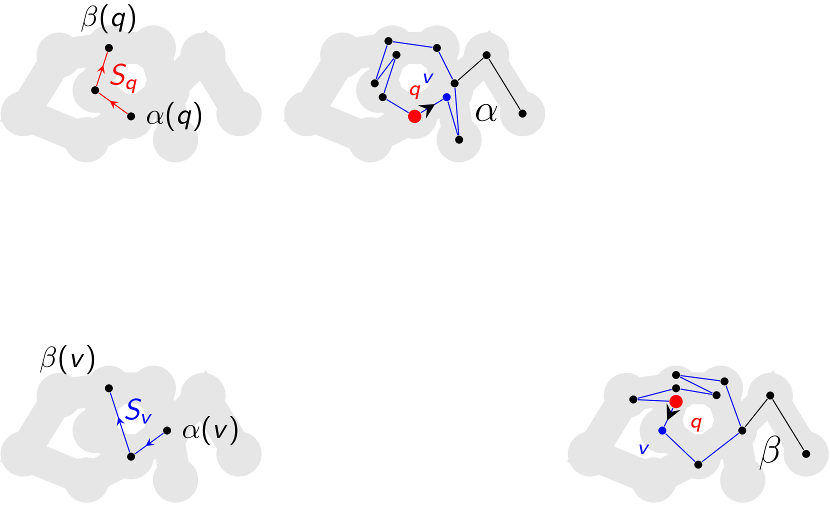
Main algorithmic idea for H-recoloring

The moves of one vertex determines the moves of all other vertices.





Example for H-Recoloring



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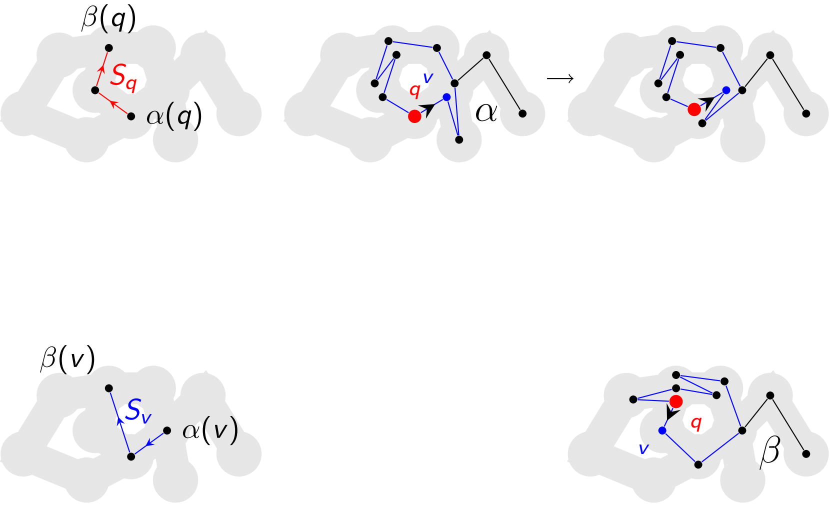

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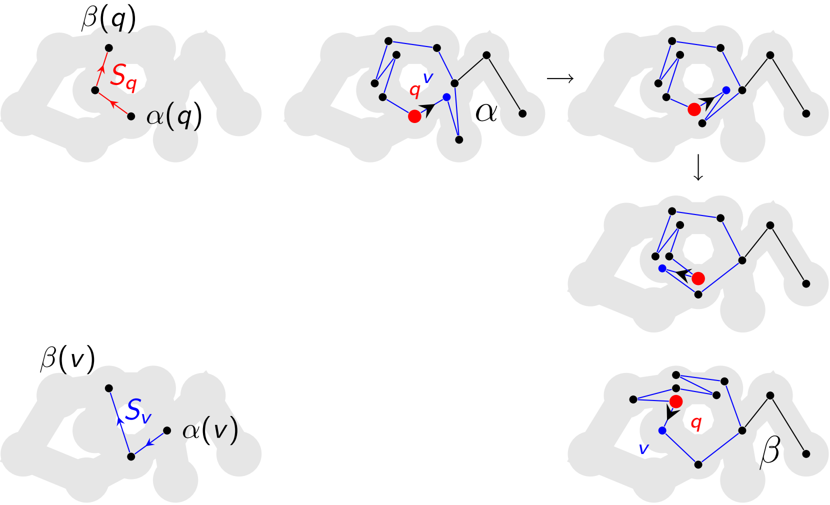

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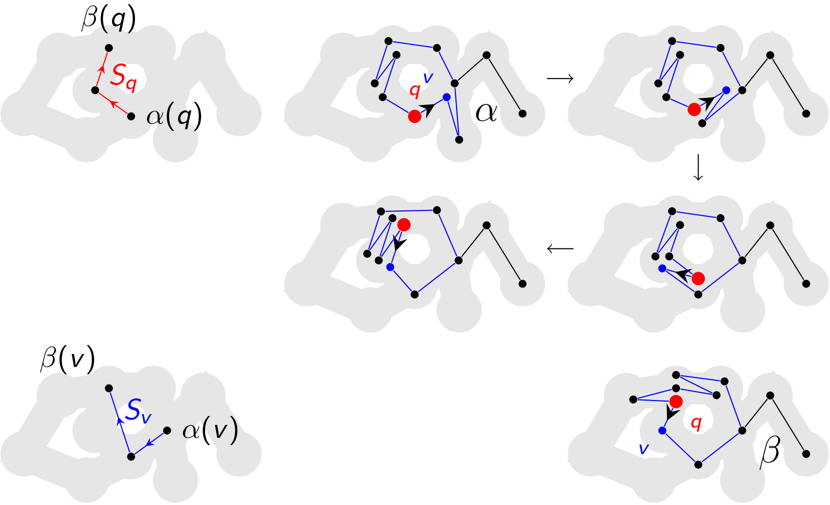

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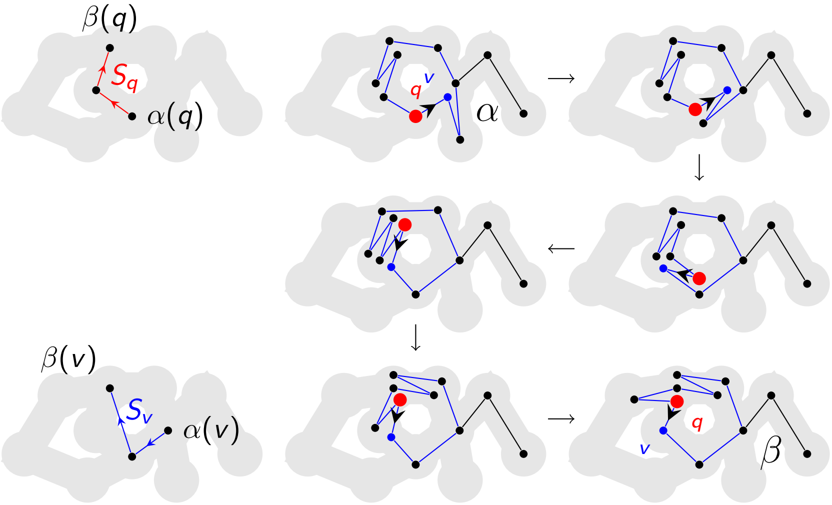


The H-mixing problem

Definition

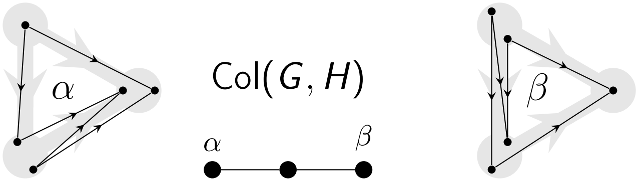
Col(G, H)

Vertices: Homomorphisms $G \to H$

Edges: single vertex recoloring.

Instance: A graph G.

Question: Is Col(H) connected?



Results (4)

H-Mixing is co-NP complete if:

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- can still be NP-hard if H is an orientation of a tree.

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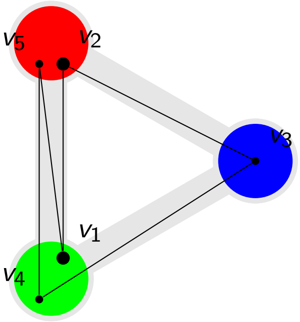
Remark

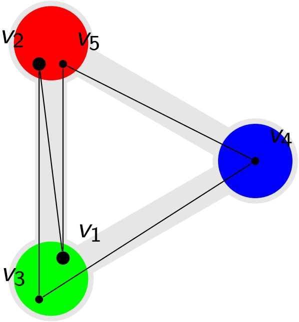
The complexity of K_k-mixing is still unknown for $k \geq 4$.

Topology for H-mixing [KLS 22]

Key idea

If $\alpha\colon G \to H$ and $\beta\colon G \to H$ wind a cycle of G in different ways then G is not H-mixing.





Topology for H-mixing [KLS 22]

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We say that $\alpha\colon G \to H$ is *flat* if α does not wind any cycle of G around a cycle of H.

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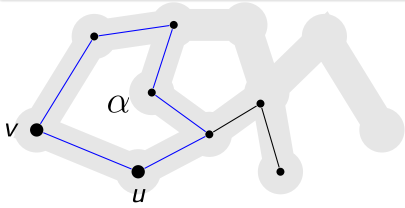

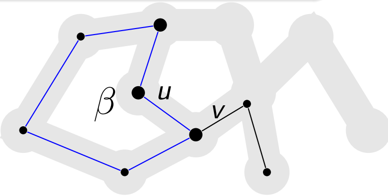

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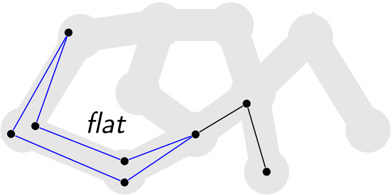
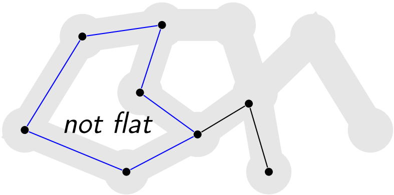
In particular, if $\alpha\colon G \to H$ is flat and not β, then G is not H-mixing.





Last (big) steps

Constructing a gadget [KLS 22]

Gadget G^a:

- There is a flat homomorphism $G^a \to H$.
- There is a non-flat homomorphism $G^a \to H$ iff $G \to K_3$.

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Adaptation of the proof

- In [KLS 22]: Underlying simplicial complex where triangles are flat (Clique complex).
- In [MSS 23+]: Realize the same topological structure as the in-neighborhood complex of an adapted gadget (and so encodes 3-coloring in the exact same way).

Thank you!