A Caro-Wei bound for induced linear forests in graphs

Gwenaël Joret Robin Petit

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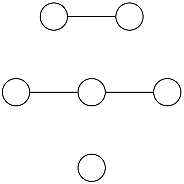
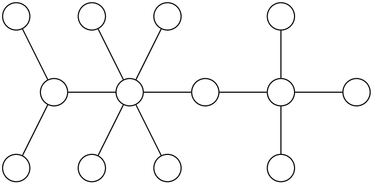
- For $W \subseteq V(G)$, $G[W]$ is the subgraph *induced* by W.
- S is an *independent* (*stable*) set in G if $E(G[S]) = \emptyset$.
- A *forest* is an acyclic graph.
- A forest is *linear* if it is a union of disjoint paths.
- A *caterpillar* is a path with an arbitrary number of leaves added.





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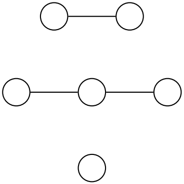

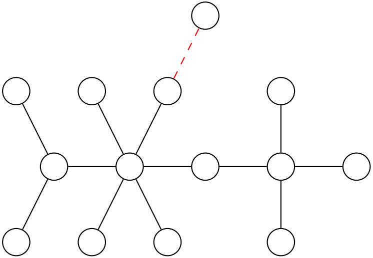

Theorem 1 (Caro-Wei, 1979-1981)

Every graph G admits an independent set of size at least:

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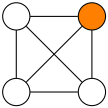

K_n

n (≥ 1) vertices of degree $n-1$.

$$\sum_{v \in V(G)} \frac{1}{d(v)+1} = n\frac{1}{n-1+1} = 1.$$

Theorem 2 (Corollary from Alon, Kahn and Seymour, 1987)

Every graph G without isolated vertices admits an induced forest of size at least:

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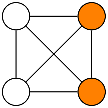

K_n

n (≥ 2) vertices of degree $n-1$.

$$\sum_{v \in V(G)} \frac{2}{d(v)+1} = n\frac{2}{n-1+1} = 2.$$

Theorem 3 (Akbari, Amanihamedani, Mousavi, Nikpey and Sheybani, 2019)

For $d \geq 1$, every d-regular graph G admits an induced linear forest of size at least $\frac{2n}{d+1}$ and this bound is tight.

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Conjecture 1 (Akbari *et al.*, 2019)

Every graph G satisfying $\delta(G) \geq 2$ admits an induced linear forest of size at least:

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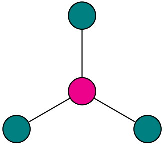

$$3 \times f(1) + 1 \times f(3) \leq 3 \quad \text{but} \quad 3 + \frac{2}{4} > 3.$$

We proved that conjecture:

Theorem 4

Let $f : \mathbb{N} \to [0,1]$ *defined as follows:*

$$f(d) = \begin{cases} 1 & \textit{if } d = 0 \\ \frac{5}{6} & \textit{if } d = 1 \\ \frac{2}{d+1} & \textit{else.} \end{cases}$$

Every graph G *admits an induced linear forest of size at least* $\sum_{v \in V(G)} f(d(v))$*, and this bound is tight.*

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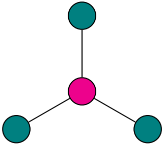

Solve $3 \times f(1) + 1 \times f(3) \leq 3$ for $f(1)$ to get $\frac{5}{6}$.

Observe that:

$$f(d) = \begin{cases} 1 & \text{if } d \leq 1 \\ \frac{2}{3} & \text{if } d = 2 \\ 0 & \text{if } d \geq 3 \end{cases}$$

is also a bound that is tight.

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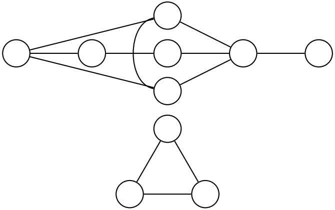

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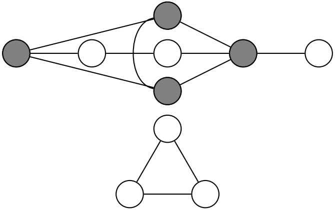

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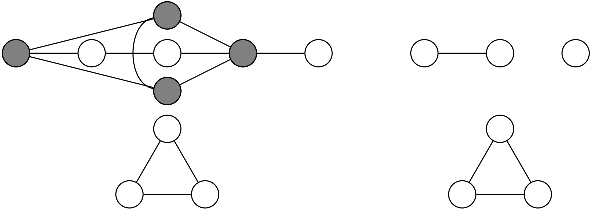

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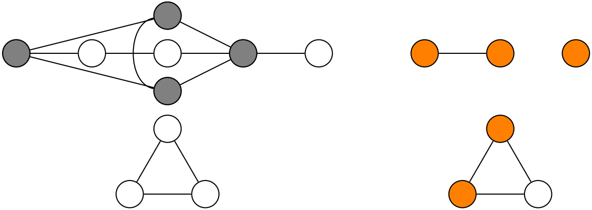

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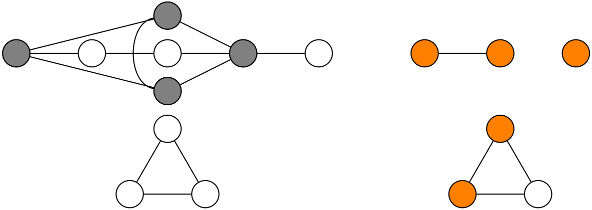

What about $\frac{5}{6} < f(1) < 1$? We can fill the gap!

Theorem 5

For every $\varepsilon \in [0, 1/6]$, *let* $f_\varepsilon : \mathbb{N} \to [0, 1]$ *defined as follows:*

$$f_\varepsilon(d) = \begin{cases} 1 & \text{if } d = 0 \\ 1 - \varepsilon & \text{if } d = 1 \\ \frac{2}{3} & \text{if } d = 2 \\ \min\{3\varepsilon, \frac{2}{d+1}\} & \text{if } d \geq 3. \end{cases}$$

Every graph G *admits an induced linear forest of size at least* $\sum_{v \in V(G)} f_\varepsilon(d(v))$, *and this bound is tight. Furthermore:*

- *The optimal* ε *on a given graph can be computed easily.*
- *These functions entirely characterise the lower bounds.*

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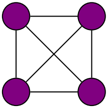

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Then replace each original vertex from the clique by a $K_{1,3}$.

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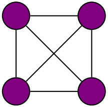

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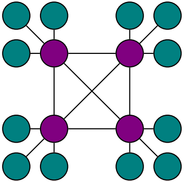

$$3 \times n \times f(1) + n \times f(n+2) \leq 3 \times n.$$
$$n \times f(n+2) \leq 3 \times n \times \varepsilon$$
$$f(n+2) \leq 3 \times \varepsilon$$

Theorem 2 (Alon, Kahn and Seymour, 1987)

Every graph G without isolated vertices admits an induced forest of size at least:

$$\sum_{v \in V(G)} \frac{2}{d(v)+1},$$

and this bound is tight.

Theorem 6

Every graph G without isolated vertices admits an induced forest of caterpillars of size at least:

$$\sum_{v \in V(G)} \frac{2}{d(v)+1},$$

and this bound is tight.

A linear forest is nothing but a caterpillar of max degree at most 2.
What “about at most k”?

A linear forest is nothing but a caterpillar of max degree at most 2. What "about at most k"?

Theorem 7

For every $k \geq 2$ and $\varepsilon \in [0, 2/(k+1)(k+2)]$, let $f_{k,\varepsilon} : \mathbb{N} \to [0,1]$ defined as follows:

$$f_{k,\varepsilon}(d) = \begin{cases} 1 & \text{if } d = 0 \\ 1 - \varepsilon & \text{if } d = 1 \\ \frac{2}{d+1} & \text{if } 2 \leq d \leq k \\ \min\{(k+1)\varepsilon, \frac{2}{d+1}\} & \text{if } d \geq k+1. \end{cases}$$

Every graph G admits an induced forest of caterpillars of degree at most k of size at least $\sum_{v \in V(G)} f_{k,\varepsilon}(d(v))$, and this bound is tight. Again, the optimal ε can be easily computed, and these functions entirely characterise the lower bounds.

1. Caro-Wei for independent sets (1979-1981).
2. Alon *et al.* for induced forests (1987).
3. Extension to caterpillars with the same bound.
4. Akbari *et al.* for induced *linear* forests in *regular* graphs (2019).
5. Characterisation of all the lower bounds for induced linear forests in all graphs.
6. Generalisation of this characterisation to forests of caterpillars of bounded degree.

Thanks for your attention! Any questions?

Choice of an optimal ε in Theorem 5:

- Note N_k the number of vertices with $d(v) = k$.
- If $3\sum_{d=3}^{\Delta(G)} N_d < N_1$, define $\varepsilon^*(G) = 0$.
- Else, let $D^*(G)$ be the smallest integer D such that $3\sum_{d=3}^{D} N_d \geq N_1$, and define $\varepsilon^*(G) = \frac{2}{3(D^*(G)+1)}$.

Then for every $\varepsilon \in [0, 1/6]$:

$$\sum_{v \in V(G)} f_{\varepsilon^*(G)}(d(v)) \geq \sum_{v \in V(G)} f_\varepsilon(d(v)).$$

Proof of Theorem 5: by the following Theorem.

Theorem 8

For a given graph G, define $C_G : V(G) \to [0,1]$ as follows:

$$C_G(v) = \begin{cases} 1 & \text{if } d(v) = 0 \text{ or } d(v) = 1 \text{ and } d(w) \leq 2 \\ 1 - \frac{2}{3(d(w)+1)} & \text{if } d(v) = 1 \text{ and } d(w) \geq 3 \\ \frac{2}{d(v)+1} & \text{if } d(v) \geq 2. \end{cases}$$

Every graph G admits an induced linear forest of size at least $\sum_{v \in V(G)} C_G(v)$.

Idea of proof of Theorem 8:

1. Remove every vertex with ≥ 3 leaves.
2. Partition $V(G)$ based on the number of leaves.
3. Remove (temporarily) the leaves.
4. Find a specific induced linear forest in what is left (bound on the degree).
5. Put back the leaves.
6. Tadaaaa.

Step 4 is the "hard one": the ABC lemma.

Theorem 9

For every $\varepsilon \in [0, 1/6]$, *let* $f_\varepsilon : \mathbb{N} \to [0, 1]$ *defined as follows:*

$$f(d) = \begin{cases} 1 & \text{if } d = 0 \\ 1 - \varepsilon & \text{if } d = 1 \\ \min\{\frac{3}{5}, \frac{1}{2} + \varepsilon\} & \text{if } d = 2 \\ \min\{\frac{2}{d+1}, \frac{1}{d} + \varepsilon\} & \text{if } d \geq 3. \end{cases}$$

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Theorem 10 (Alon, Kahn and Seymour, 1987)

For every integer k, every graph G admits an induced subgraph $H \leq G$ that is k-degenerate of size at least:

$$\sum_{v \in V(G)} \min\left\{1, \frac{k+1}{d(v)+1}\right\},$$

and this bound is tight.

Where a graph H is k-degenerate if every subgraph of H has a vertex of degree at most k.