Skipless Chain Decompositions & Improved Poset Saturation Bounds

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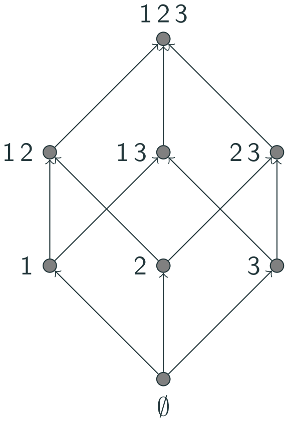
Hugo Jacob *ENS Paris-Saclay*
Tom Johnston *University of Bristol*

Boolean lattice

The Boolean lattice of dimension n:

- elements: $2^{[n]} = \mathcal{P}(\{1, \ldots, n\})$
- relation: $\subseteq$



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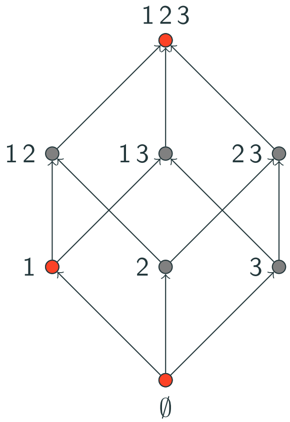
An antichain is a set system where every pair of elements is incomparable.



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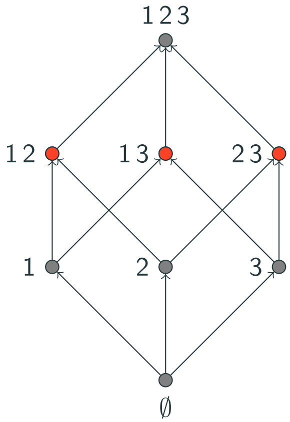

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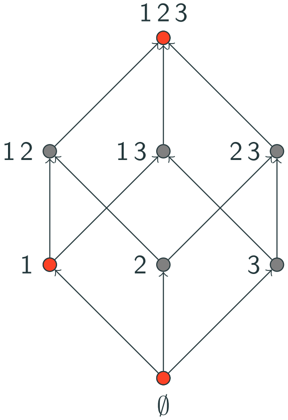
A chain $C = \{C_1 \subsetneq C_2 \subsetneq \ldots \subsetneq C_k\} \subseteq P$ is skipless in P if for all $i \in [k-1]$, there is no $X \in P$ with $C_i \subsetneq X \subsetneq C_{i+1}$.



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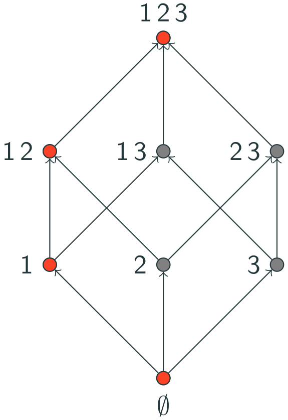

Chains in the hypercube

Theorem (Dilworth 1950)

For a family poset $\mathcal{P}$, the size of the largest antichain is equal to the size of smallest chain disjoint chain decomposition of $\mathcal{P}$.

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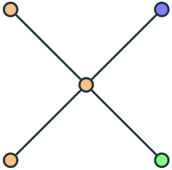

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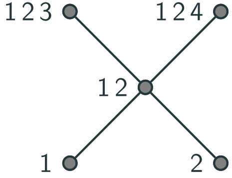

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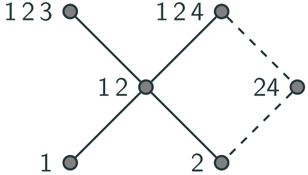

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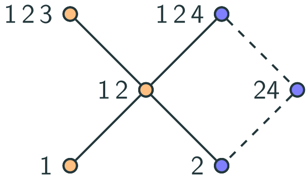

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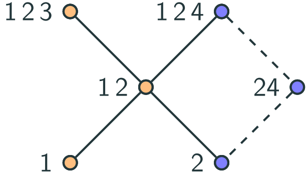

True for every poset, and every way to embed it.

Cover chains with skipless chains

Structural Theorem [B., Groenland, Jacob, Johnston, 2022+]

Any subposet $\mathcal{P}$ of $2^{[n]}$ with largest antichain of size k can be **covered** by a family of k **disjoint skipless** chains in $2^{[n]}$.

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We generalise a result of Lehman and Ron (2001) who proved the special case where all chains of the family are of size 2 and all top (resp. bottom) elements of the chain have the same size.

We generalise a result from Duffus, Howard and Leader (2019) who proved the special case where the family is convex[1].

[1] $\mathcal{F} \subseteq 2^{[n]}$ is convex if for all $X, Z \in \mathcal{F}$ and $X \subset Y \subset Z$, $Y \in \mathcal{F}$.

Sketch of the sketch of the proof

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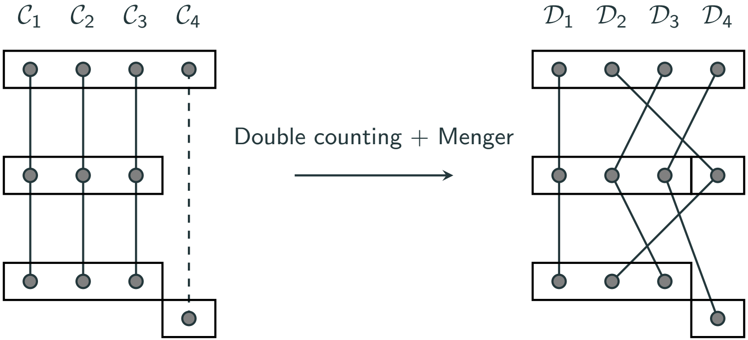

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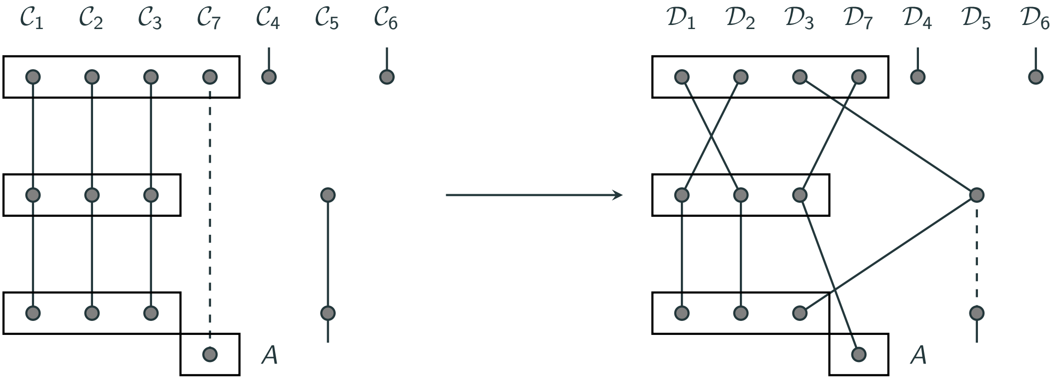

Antichain saturation

Antichain saturation

$\mathcal{F} \subseteq 2^{[n]}$, is *$k$-saturated* if:

- $\mathcal{F}$ has no antichain of size k;
- $\mathcal{F} \cup \{x\}$ has an antichain of size k for any $x \in 2^{[n]} \setminus \mathcal{F}$.

sat*$(n, k) =$ minimum $|\mathcal{F}|$ over all k-saturated families $\mathcal{F}$ in $2^{[n]}$.

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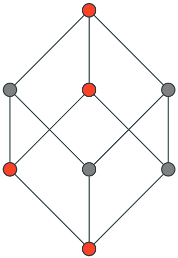
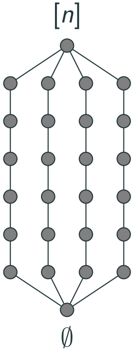
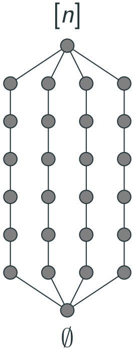


Red sets form an 2-saturated family for the hypercube $2^{[3]}$: sat*$(3, 2) \leq 4$.
Can we extend this construction to k-saturated ?



Construction: $\text{sat}^*(n,k) \leq (n-1)(k-1)+2$.

Antichain saturation



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Ferrara, Kay, Kramer, Martin, Reiniger, Smith and Sullivan (2017).

k	2	3	4
$\text{sat}^*(k,n)$	$n+1$	$2n$	$3n-1$

Conjecture (FKKMRSS): $\forall k \geq 2$, $\text{sat}^*(n,k) \sim n(k-1)$ as $n \to \infty$.

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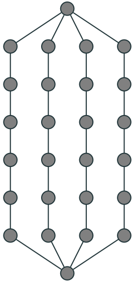

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Conjecture (Đanković and Ivan): $\forall k \ge 2$, $\text{sat}^*(n,k) \ge n(k-1) - C_k$.

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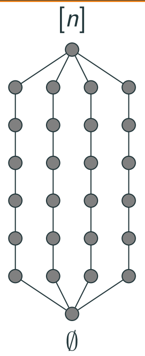

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There is a half page proof of both conjecture with our decomposition lemma!

Exact values

$\nu(\mathcal{F}) \to$ the size of the maximum matching from $\mathcal{F}$ to its shadow $\partial\mathcal{F}$.

$\mathcal{C}(m, t) \to$ initial segment of colex of size m on layer t.

Define the sequence $c_{\lfloor \ell/2 \rfloor} = k - 1$, and for $0 \leq t < \lfloor \ell/2 \rfloor$, let $c_t = \nu\left(\mathcal{C}(c_{t+1}, t+1)\right)$.

B, Groenland, Jacob and Johnston (2023+)

For $n \geq 2\log(k) + 1$,

$$\text{sat}^*(n, k) = 2 \sum_{t=0}^{\lfloor \ell/2 \rfloor} c_t + (k-1)(n - 1 - 2\lfloor \ell/2 \rfloor).$$

The lower bound still holds for $n \geq \ell$ (and $\text{sat}^*(n, k) = 2^n$ for $n < \ell$).

General saturation

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Definition

$\mathcal{F} \subseteq 2^{[n]}$ a set system is $\mathcal{P}$-saturated if:

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Theorem (Morrison, Noel and Scott 2014;

$$\leq \text{sat}^*(n, C_k) \leq 2^{0.98k}$$

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Theorem (Morrison, Noel and Scott 2014; Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi, Patkós 2011)

$$2^{(k-3)/2} \leq \text{sat}^*(n, C_k) \leq 2^{0.98k}$$

Table

poset P	$\mathbf{sat}(n,P)$	$\mathbf{sat}^*(n,P)$	
C_2, chain	$=1$	$=1$	
A_2, antichain	$=1$	$=n+1$	
C_3, chain	$=2$	$=2$	
C_2+C_1, chain and single	$=2$	$=4$	case analysis
$\vee$ fork (or $\wedge$)	$=2$	$=n+1$	[F7]
A_3, antichain	$=2$	$=3n-1$	[F7]
C_4, chain	$=4$	$=4$	[G6]
$\vee_3$, fork with three tines	$=3$	$\geq \log_2 n$	[F7]
$\Diamond$, diamond	$=3$	$\geq \sqrt{n}$ $\leq n+1$	[MSW] [F7]
$\Diamond^-$, diamond minus an edge	$=3$	$=4$	case analysis
$\bowtie$, butterfly	$=4$	$\geq n+1$ $\leq 6n-10$	[I] [Thm 3.16]
Y	$=3$	$\geq \log_2 n$	[Thm. 3.6]
N	$=3$	$\geq \sqrt{n}$ $\leq 2n$	[I] [F7]
$2C_2$	$=3$	$\geq n+2$ $\leq 2n$	[Thm. 3.11] [Prop. 3.9]

Figure 1: Table from Keszegh, Lemons, Martin, Pálvölgyi and Patkós 2022

Table

$C_3 + C_1$, chain and single	$= 3$	≤ 8	[Prop. 3.18]
$\vee + 1$, fork and single	$= 3$	$\geq \log_2 n$	[F7]
$C_2 + A_2$	$= 3$	≤ 8	[Prop. 3.18]
A_4, antichain	$= 3$	$\geq 3n - 1$ $\leq 4n + 2$	[F7] [F7]
C_5, chain	$= 8$	$= 8$	[G6]+[MNS]
C_6, chain	$= 16$	$= 16$	[G6]+[MNS]
C_k, chain ($k \geq 7$)	$\geq 2^{(k-3)/2}$ $\leq 2^{0.98k}$	$\geq 2^{(k-3)/2}$ $\leq 2^{0.98k}$	[G6] [MNS]
A_k, antichain	$= k - 1$	$\geq \left(1 - \frac{1}{\log_2 k}\right) \frac{k}{\log_2 k} n$ $\leq kn - k - \frac{1}{2} \log_2 k + O(1)$	[MSW] [F7]
$3C_2$	$= 5$	≤ 14	[Prop. 3.13]
$5C_2$	$= 9$	≤ 42	[Prop. 3.18]
$7C_2$	$= 13$	≤ 60	[Prop. 3.18]
any poset on k elements	$\leq 2^{k-2}$	—	[Thm. 1.1]
UCTP (def. in Section 3.2)	$O(1)$	$\geq \log_2 n$	[F7]
UCTP with top chain	$O(1)$	$\geq \log_2 n$	[Thm. 3.6]
chain + shallower	$O(1)$	$O(1)$	[Thm. 3.8]

Figure 2: Table from Keszegh, Lemons, Martin, Pálvölgyi and Patkós 2022

General bounds

Very recently, a general lower bound has been shown.

Theorem (Freschi, Piga, Sharifzadeh and Treglown 2023)

For any poset P either $\text{sat}^*(n, P) \geq 2\sqrt{n} - 2$ *or* $\text{sat}^*(n, P) = O_P(1)$.

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Theorem (B., Groenland, Ivan, Johnston, 2023+)

For any poset P, $\text{sat}^*(n,P) \leq n^{|P|^2}$.

Open question

Conjecture

For every poset $\mathcal{P}$, either $\text{sat}^*(n, \mathcal{P}) = O_{\mathcal{P}}(1)$ or $\text{sat}^*(n, \mathcal{P}) = \Theta_{\mathcal{P}}(n)$.

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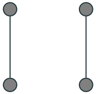
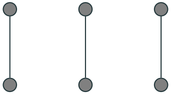


$\text{sat}^*(2C_2, n) \geq n$



$\text{sat}^*(3C_2, n) \leq 14$

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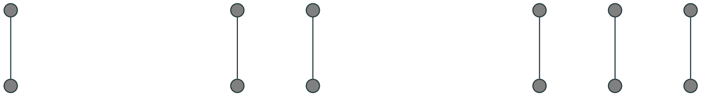

$\text{sat}^*(C_2, n) = 1$ $\quad$ $\text{sat}^*(2C_2, n) \geq n$ $\quad$ $\text{sat}^*(3C_2, n) \leq 14$

Thank you!

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Figure 3: Table from [?]

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