Some problems and results on acyclic sets and colouring of digraphs

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Joint work with Ararat Harutyunyan[1] and Colin McDiarmid[2]

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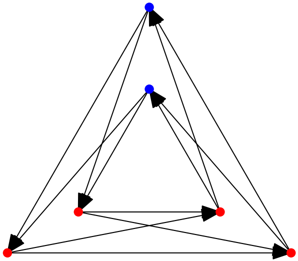
Acyclic colouring



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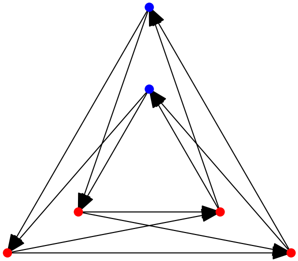

$\vec{\alpha}(D) := \max\{|S| \mid S \subseteq V(D) \text{ acyclic}\}$

$\vec{\chi}(D) := \min\{|P| \mid P \text{ partition of } V(D) \text{ into acyclic sets}\}$

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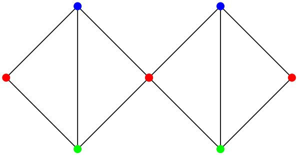

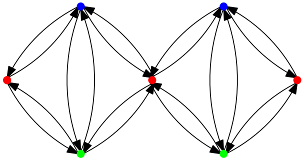

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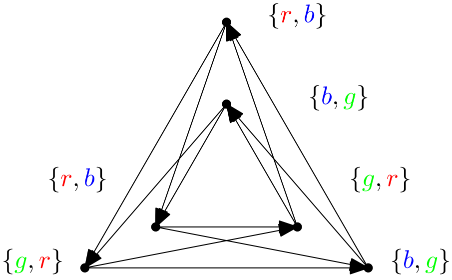

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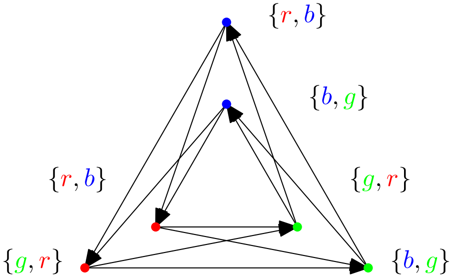

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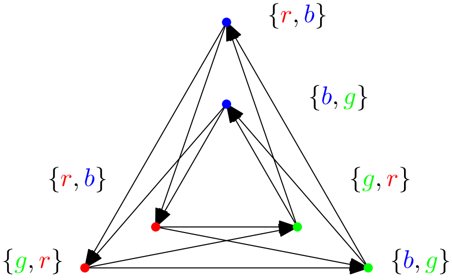
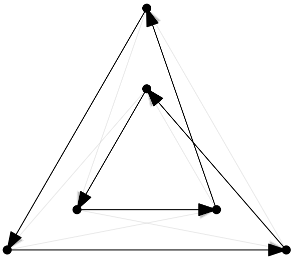


$\vec{\chi}_\ell(D) :=$ min$\{k \mid$ every assignment of k-lists admits an acyclic colouring$\}$

$\vec{\chi}(D) \leq \vec{\chi}_\ell(D)$

Circumference bounds



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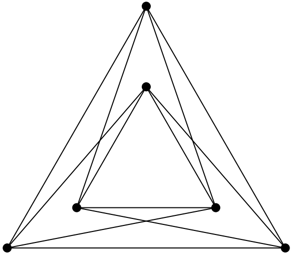
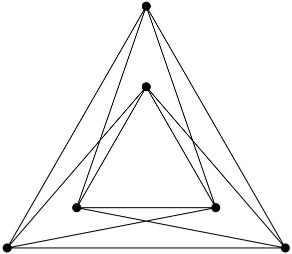


Theorem (Bondy, 1976)

Let G be an undirected graph. If G has a strongly connected orientation with circumference s, then $\chi(G) \leq s$.

Circumference bounds



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Theorem (Chen, Ma and Wang, 2015)

Let $k, r \in \mathbb{Z}$ with $k \geq 2$. If a digraph D has no directed cycle of length r modulo k, then $\vec{\chi}(D) \leq k$.

Circumference bounds

Theorem

Let D be a digraph in which the set of all (directed) cycle lengths has size k. Then $\vec{\chi}_\ell(D) \leq k + 1$.

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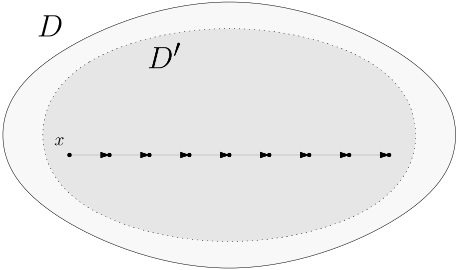

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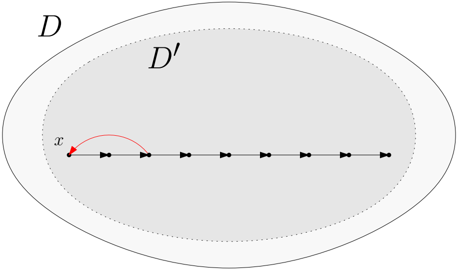

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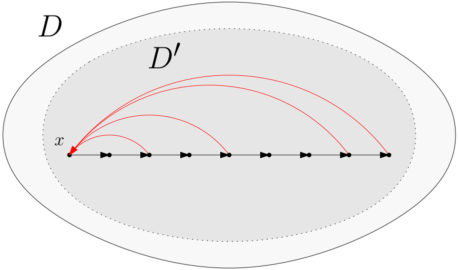

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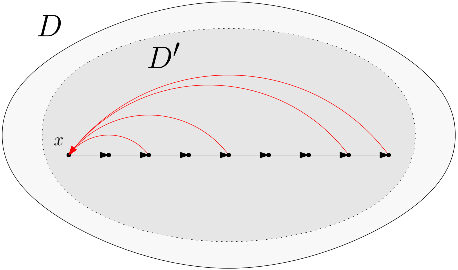

Corollary

Let D be an oriented graph with circumference s. Then $\vec{\chi}_\ell(D) \leq s-1$.

Circumference bounds

Theorem (cf. Cordero-Michel and Galeana-Sánchez, 2019)

Let D be a digraph with circumference s and digirth g. Then $\vec{\chi}(D) \leq \lceil s/(g-1) \rceil$.

Circumference bounds

Theorem (cf. Cordero-Michel and Galeana-Sánchez, 2019)

Let D be a digraph with circumference s and digirth g. Then $\vec{\chi}(D) \leq \lceil s/(g-1) \rceil$.

Question

Let D be an oriented graph with circumference s. Is it true that $\vec{\chi}(D) = O(s/\ln s)$?

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Theorem

Let T be a tournament with circumference s. Then $\vec{\chi}_\ell(T) \leq (1+o(1))s/\log_2 s$ as $s \to \infty$.

Degree bounds

Conjecture (Aharoni, Berger and Kfir, 2008)

Let D be an n-vertex oriented graph with average outdegree d^+. Then, $\vec{\alpha}(D) \geq (1 + o(1))n \log_2 d^+/d^+$.

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Theorem (Shearer, 1982)

Let G be an n-vertex triangle-free graph with average degree d. Then, $\alpha(G) \geq (1 + o(1))n \ln d / d$.

Degree bounds

Theorem (Spencer and Subramanian, 2008)

Let $\mathcal{D}(n,p)$ be the random oriented graph on n vertices obtained by choosing each of the $\binom{n}{2}$ possible edges independently with probability $2p$ and then orienting each of the chosen edges independently with probability $1/2$. Then,

$$\vec{\alpha}(\mathcal{D}(n,p)) = \frac{2\ln(np)}{-\ln(1-p)}(1+o(1))$$

asymptotically almost surely, provided that $p \geq W/n$, where W is a fixed constant.

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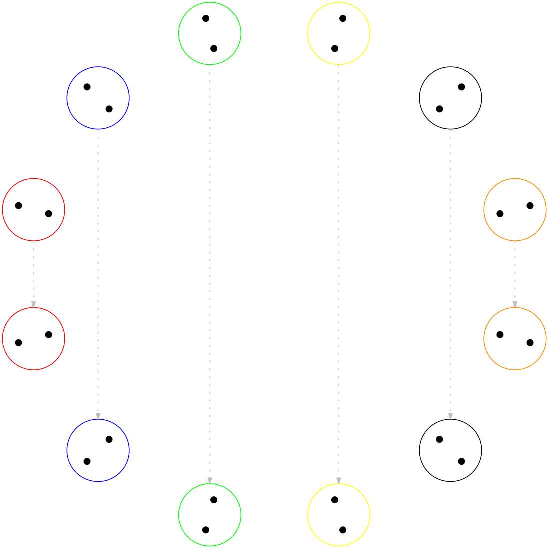
Theorem

Let $\mathbf{D}(n,d)$ be the uniform random d-regular digraph on n vertices. If $d \geq 2$, then $\vec{\alpha}(\mathbf{D}(n,d)) = \Theta(n \ln d/d)$ asymptotically almost surely.

Degree bounds



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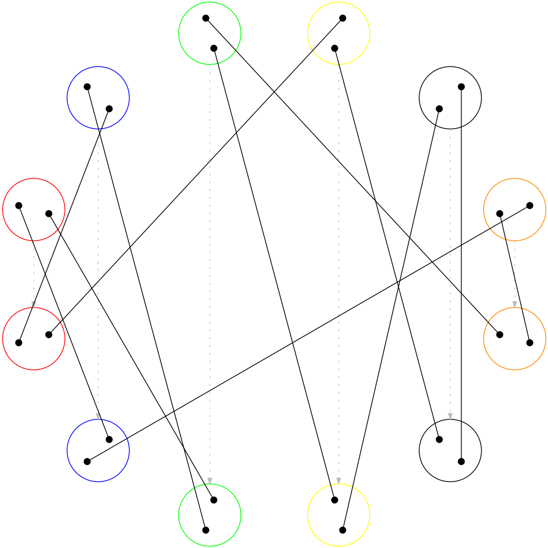

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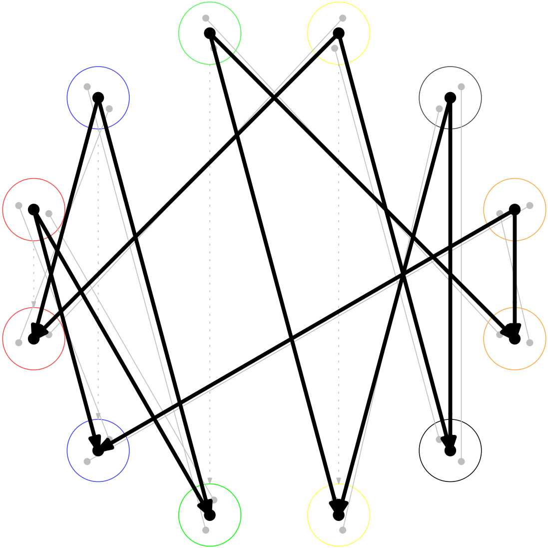

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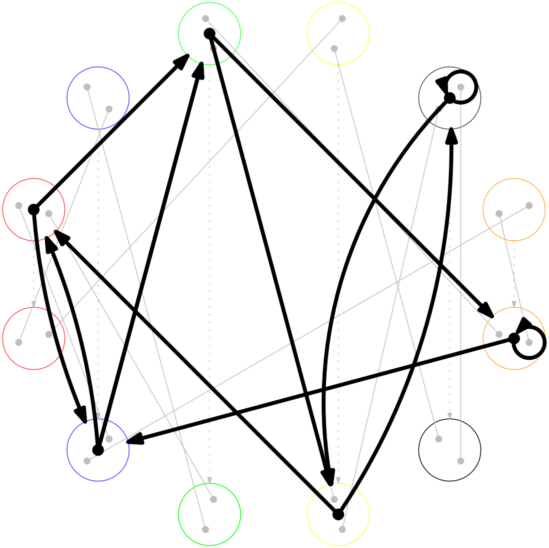

Degree bounds

Theorem

$\vec{T}_3$-free digraphs satisfy the *ABK* conjecture, up to multiplication by a constant.

Question

Do $\vec{C}_3$-free digraphs satisfy the *ABK* conjecture (up to multiplication by a constant)?