On/off Graphs

Antoine Castillon\textsuperscript{1,2}, J. Baste, C. Dhaenens\textsuperscript{1}, M. Haddad, H. Seba\textsuperscript{2}

1. CRIStAL Lille, 2. LIRIS Lyon

JGA 2023
Whac-A-Mole
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We define an on/off graph as a tuple \((G, \alpha, \omega)\) where \(G = (V, E)\) is a graph and \(\alpha : V \to \mathbb{N}^*, \omega : V \to \mathbb{N}^*\) such that \(\forall v \in V, \alpha(v) \leq \omega(v)\).

We note:
- Active vertices \(V_t = \{v \in V \mid \alpha(v) \leq t < \omega(v)\}\).
- Active subgraph \(G_t = G[V_t]\).
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Problem (On/off-\(\pi\) problem)

- **Input**: \(G = (V, E)\) a graph, \(\alpha : V \rightarrow \mathbb{N}\).
- **Output**: \(\omega : V \rightarrow \mathbb{N}\) s.t.
  - \((G, \alpha, \omega)\) is an on/off graph,
  - \(|\omega^{-1}(t)| \leq \Omega\), \(\Omega\) being a fixed constant,
  - \(G_t\) verifies \(\pi\).
Problem (On/off-Clique problem)

- **Input:** $G = (V, E)$ a graph, $\alpha : V \to \mathbb{N}$.
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On/off Problems

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Classical complexity

- On/off-Clique is NP-complete, for all constant $\Omega \geq 1$. 
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Given $\pi$ a graph property s.t.
- $\pi$ is non trivial,
- $\pi$ is hereditary,
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**Theorem (Yannakakis 1978)**

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**Theorem (Adapted)**

*The on/off-$\pi$ problem is NP-complete, for all constant $\Omega \geq 1$.***
Parameters specific to on/off graphs:

- $T = \max_{v \in V} \alpha(v)$,
- $A = \max_{0 \leq t \leq T} |\alpha^{-1}(t)|$,
- $\Omega = \max_{0 \leq t \leq T} |\omega^{-1}(t)|$.
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Graph parameters:

- $\Delta$, $h$-index, degeneracy, ...
- Widths: $tw$, $cw$, $pw$, ...
Parameterized Complexity

Parameters specific to on/off graphs:

- $T = \max_{v \in V} \alpha(v)$,
- $A = \max_{0 \leq t \leq T} |\alpha^{-1}(t)|$, paraNP (NP-hard with $A \geq 2$)
- $\Omega = \max_{0 \leq t \leq T} |\omega^{-1}(t)|$, paraNP (NP-hard with $\Omega \geq 1$)

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Generic algorithm

Enumerating all subgraphs of $G$ verifying $\pi \implies$ solving on/off-$\pi$ problem.
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1. **Enumerate** subgraphs verifying \( \pi \)
2. **Construct** configuration graph
3. **Output** path from \((\emptyset, 0)\) to \((\ldots, T)\)

\[
\begin{align*}
&+ \{ v \mid \alpha(v) = t \} \\
&- \{ \text{one vertex} \}
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Consequences:
- on/off-Clique is FPT when parameterized by $tw, \Delta, h, d, ...$. 

\[ +\{v \mid \alpha(v) = t\} \quad \text{and} \quad -\{\text{one vertex}\} \]
On/off-Independent Set

→ Solvable in $T^{tw} \cdot 2^T \cdot n^{O(1)} \implies$ FPT in $T + tw$. 
On/off-Independent Set

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On/off-Independent Set is NP-hard even when restricted to paths.
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Theorem

On/off-Independent Set is NP-hard even when restricted to paths.

Consequences:

- On/off-Independent Set is paraNP-hard when parameterized by: $tw, cw, pw, \Delta, h, d, \ldots$. 
On/off-Independent Set on paths

Use the $\alpha$ function to artificially link the vertices:
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$$\alpha \leq t_0 \quad \alpha = t_0 + 2 \quad \alpha = t_0 + 1$$

Now $u$ and $v$ are incompatible.
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Now $u$ and $v$ are incompatible.
On/off-Independent Set on paths

Using the previous gadget we can “remove” cycles:
On/off-Independent Set on paths

Some variants of the gadget:
On/off-Independent Set on paths

Some variants of the gadget:

\[ u \quad t_0 \quad u \]

\[ u_1 \leftrightarrow u_2 \]

clones
On/off-Independent Set on paths

Some variants of the gadget:

- Clones: $u_1 \leftrightarrow u_2$
- Hyperedges: $\{u, v, w\}$

Diagram:

- Two nodes $u$ connected by a dashed line $t_0$.
- Clones $u_1 \leftrightarrow u_2$.
- Hyperedges forming a triangle with $u, v, w$. 
To sum up

→ On/off-$\pi$ problem is NP-complete.

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<tr>
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The FPT(T + tw) algorithm is optimal for tw what about for T?
→ On/off-$\pi$ problem is NP-complete.

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→ We believe that the result obtained for Independent Set can be extended to half of the properties.
→ On/off-$\pi$ problem is NP-complete.

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→ We believe that the result obtained for Independent Set can be extended to half of the properties.
→ Game theory: Maker-Breaker, 1st player activate the vertices and 2nd player deactivate them.
Thank You!