On/off Graphs

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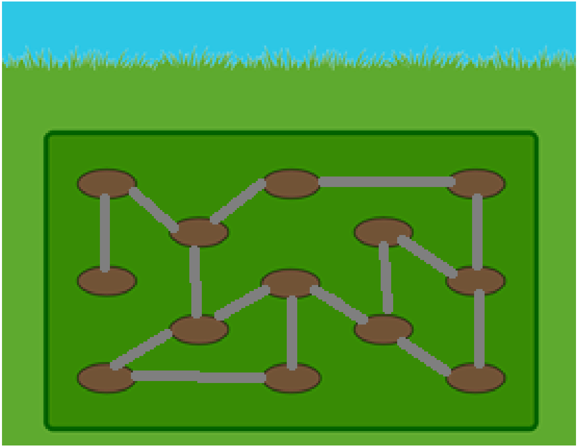
Whac-A-Mole



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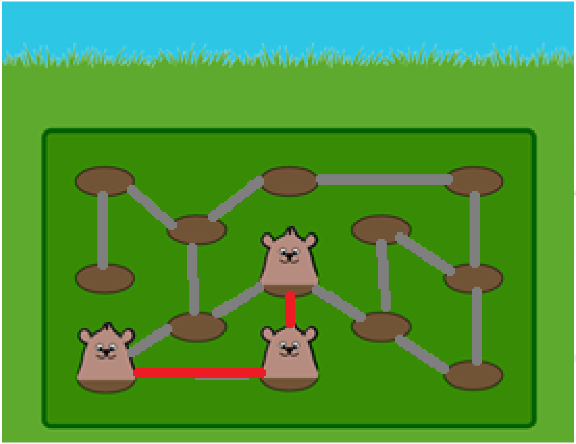
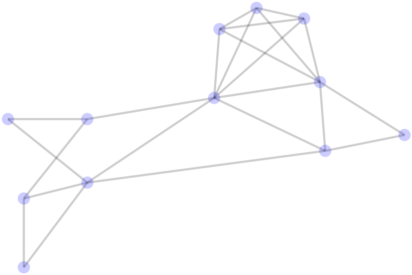


Concept

Definition (On/off Graph)

We define an on/off graph as a tuple (G, α, ω) where $G = (V, E)$ is a graph and $\alpha : V \to \mathbb{N}^*$, $\omega : V \to \mathbb{N}^*$ such that $\forall v \in V, \alpha(v) \leq \omega(v)$.



We note:

- Active vertices $V_t = \{v \in V \mid \alpha(v) \leq t < \omega(v)\}$.
- Active subgraph $G_t = G[V_t]$.

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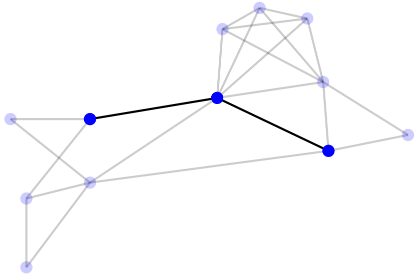

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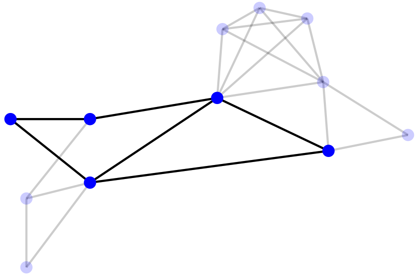

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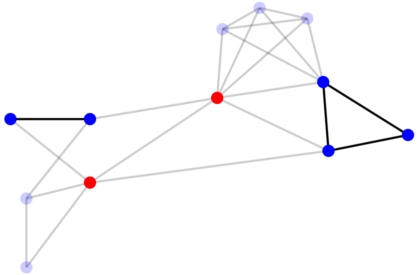

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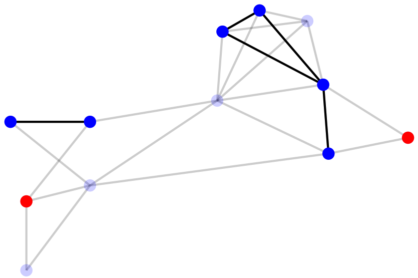

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On/off Problems

Problem (On/off-π problem)

- **Input:** $G = (V, E)$ a graph, $\alpha : V \to \mathbb{N}$.
- **Output:** $\omega : V \to \mathbb{N}$ s.t.
  - (G, α, ω) is an on/off graph,
  - $|\omega^{-1}(t)| \leq \Omega$, Ω being a fixed constant,
  - G_t verifies π.

On/off Problems

Problem (On/off-Clique problem)

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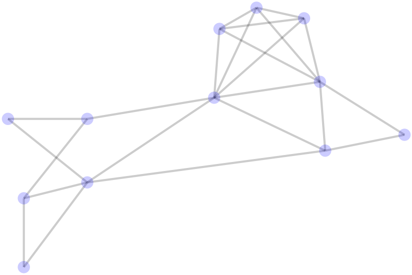

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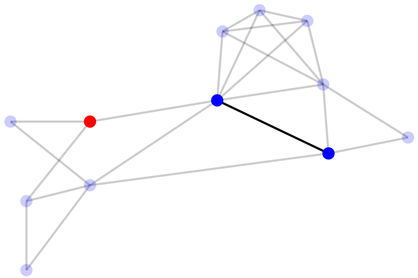

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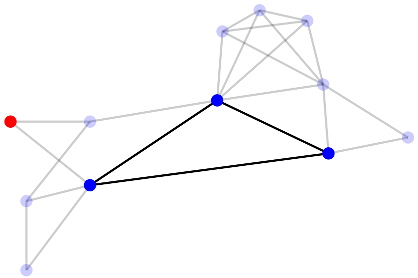

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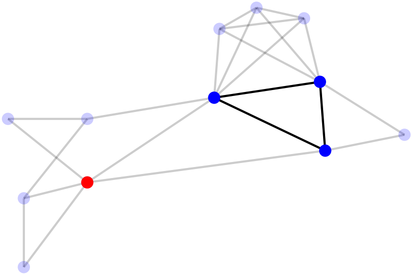

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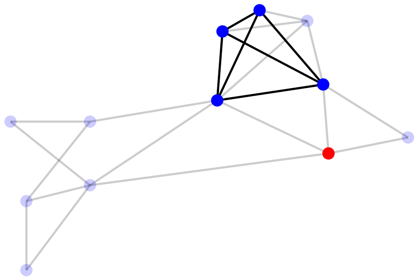

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Theorem (Adapted)

The on/off-π problem is NP-complete, for all constant $\Omega \geq 1$.

Parameterized Complexity

Parameters specific to on/off graphs:

- $T = \max_{v \in V} \alpha(v)$,
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- Widths: $tw, cw, pw, \ldots$

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- $\Omega = \max_{0 \leq t \leq T} |\omega^{-1}(t)|$, paraNP (NP-hard with $\Omega \geq 1$)

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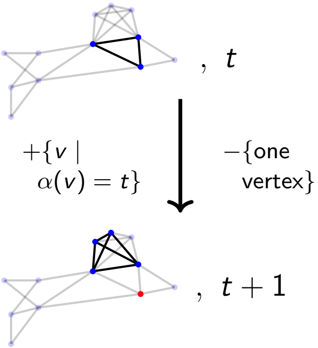

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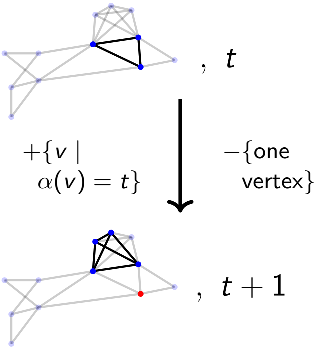

Consequences:

- on/off-Clique is FPT when parameterized by $tw, \Delta, h, d, ...$.

On/off-Independent Set

$\rightarrow$ Solvable in $T^{tw} \cdot 2^T \cdot n^{O(1)} \implies$ FPT in $T + tw$.

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Consequences:

- On/off-Independent Set is paraNP-hard when parameterized by : $tw, cw, pw, \Delta, h, d, \ldots$.

On/off-Independent Set on paths

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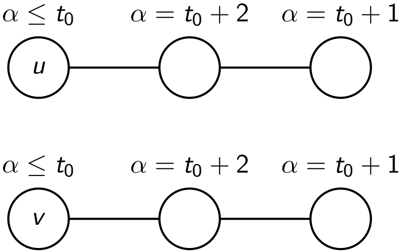

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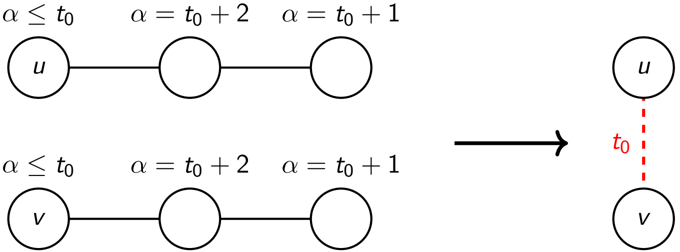

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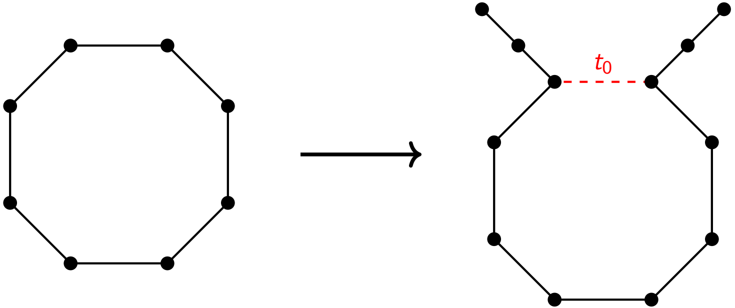
On/off-Independent Set on paths

Using the previous gadget we can “remove” cycles:



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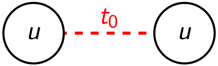

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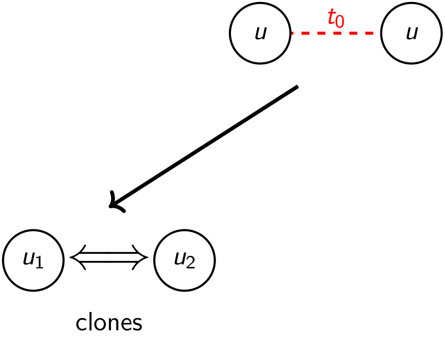

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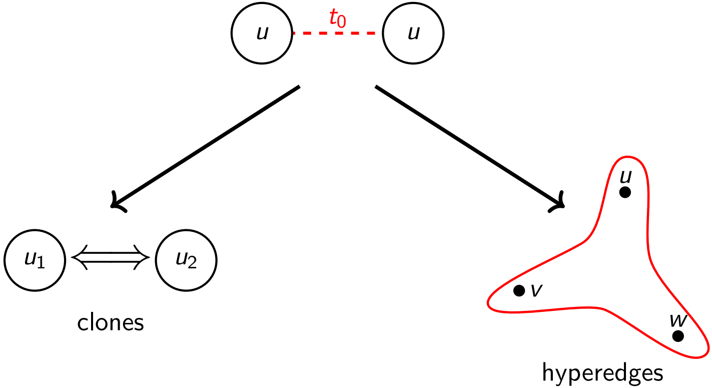

To sum up

→ On/off-π problem is NP-complete.

	Δ, h, d	tw	$T + tw$	T	A	Ω
Clique	FPT	FPT	FPT	?	paraNP	paraNP
Independant Set	paraNP	paraNP	FPT	?	paraNP	paraNP
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→ Game theory: Maker-Breaker, 1st player activate the vertices and 2nd player deactivate them.

Thank You !