Oriented trees in digraphs of high chromatic number

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joint work with
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Journées Graphes et Algorithmes
21 Novembre 2023, Lyon

Preliminaries

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- sub**di**graphs: **this talk!**

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Given H, is there some k s.t. $\chi(G) \geq k \implies G$ contains H as a subgraph?

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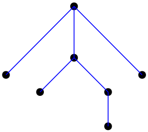
Theorem (Erdős, Hajnal '1966)
There exist graphs of arbitrarily large girth and χ.

$\hookrightarrow$ if H is not a tree, **no**.

χ-vertex trees in simple graphs

Graph G: find trees on $\chi(G)$ vertices as subgraphs, **tight**.



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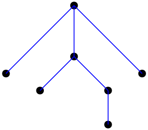

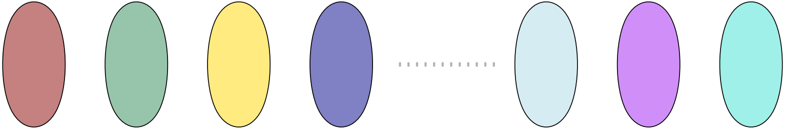

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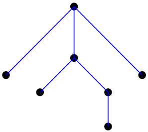

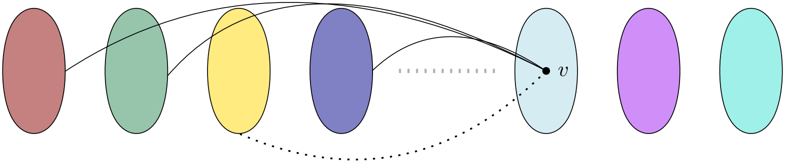

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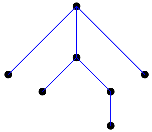

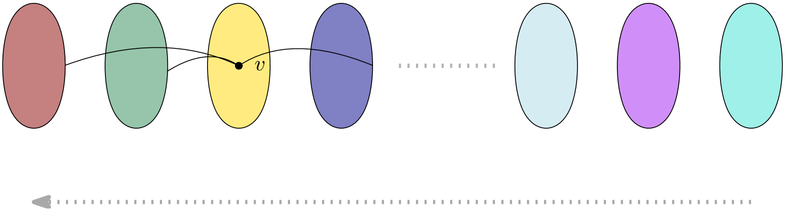

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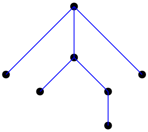

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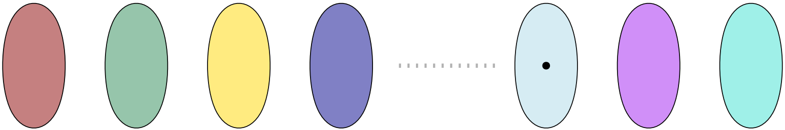

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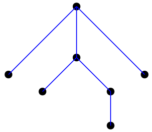

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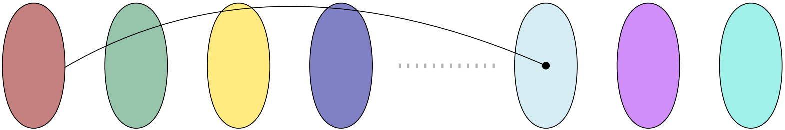

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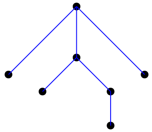

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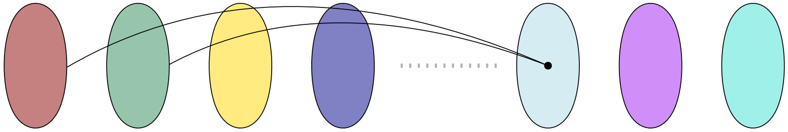

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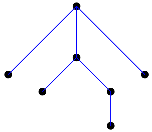

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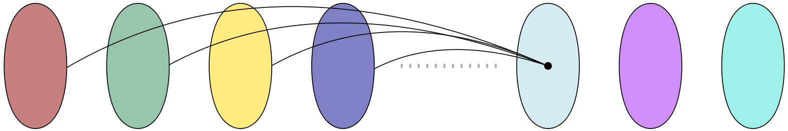

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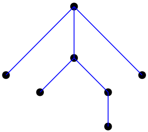

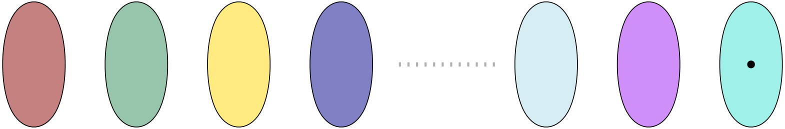

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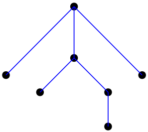

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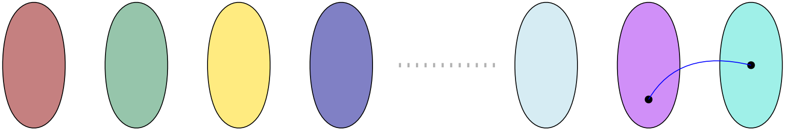

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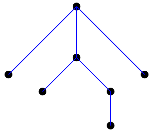

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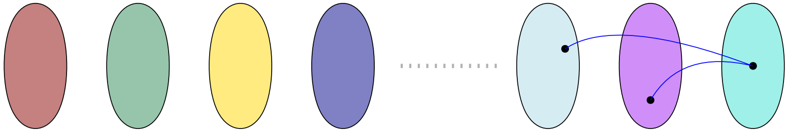

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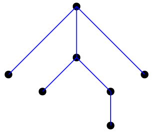

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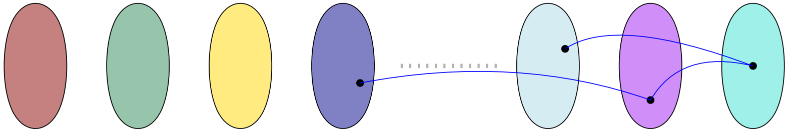

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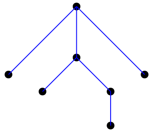

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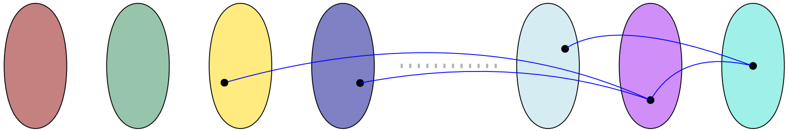

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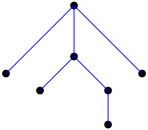

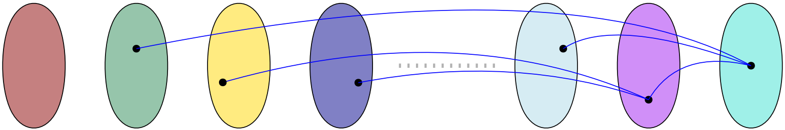

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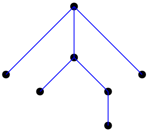

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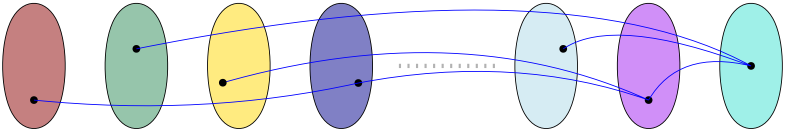

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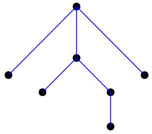

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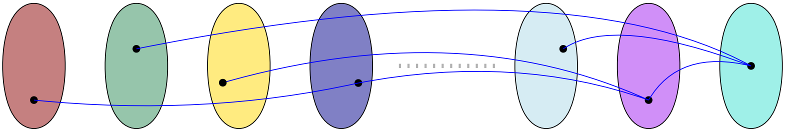

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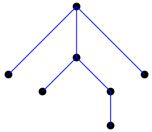

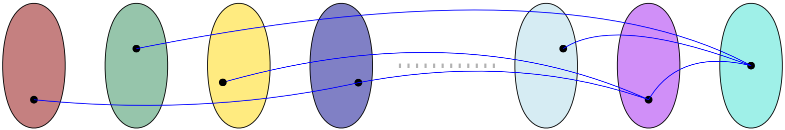

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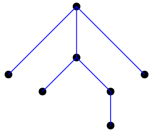

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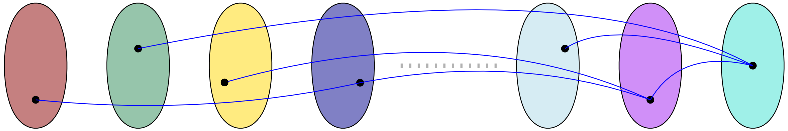

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- *This talk*: find k-vertex oriented subtrees when $\chi \geq f(k)$.

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Conjecture (Burr '80)
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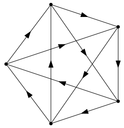

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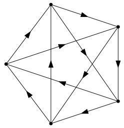

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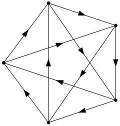

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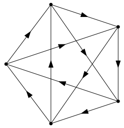

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- $\geq 10k \implies$ antidirected trees, ≤ 4 blocks paths

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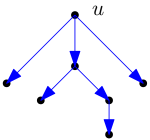

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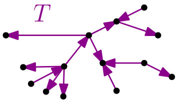

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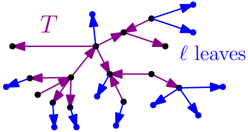

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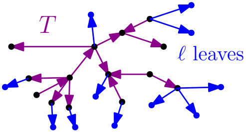

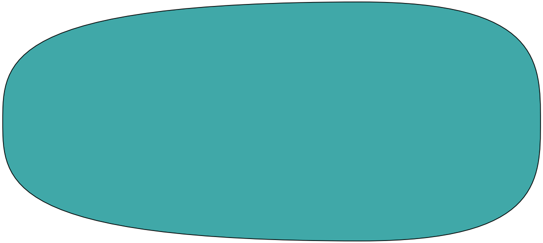

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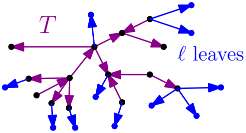

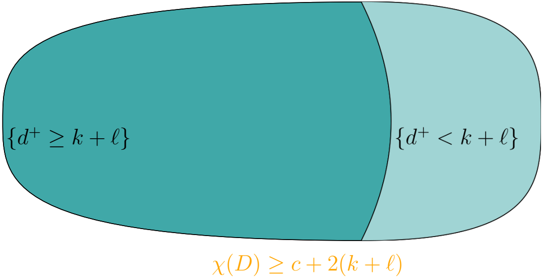

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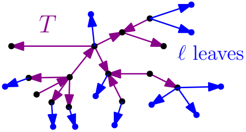

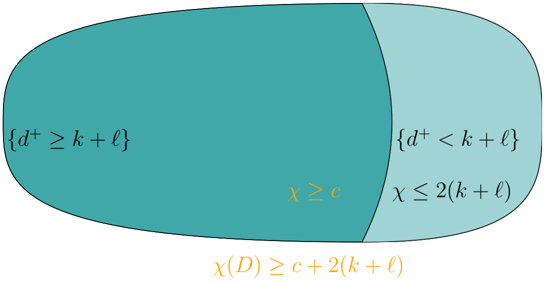

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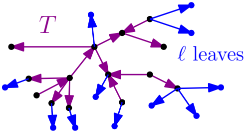

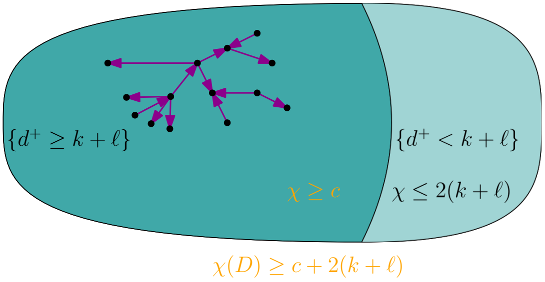

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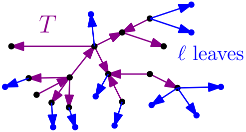

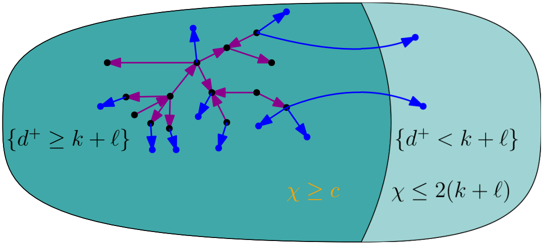

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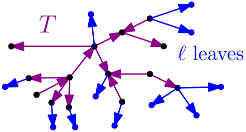

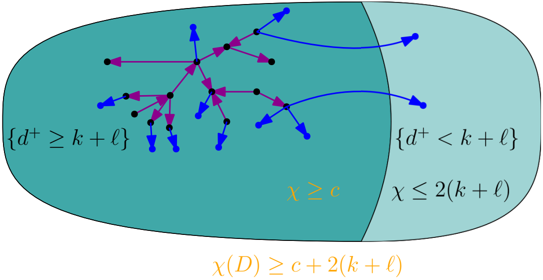

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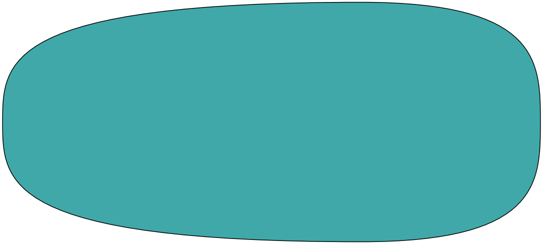

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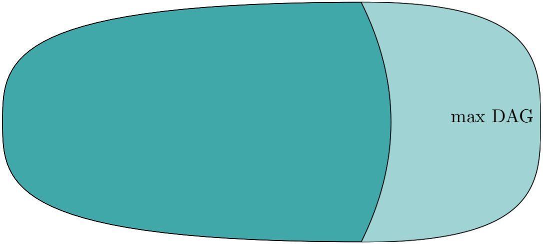

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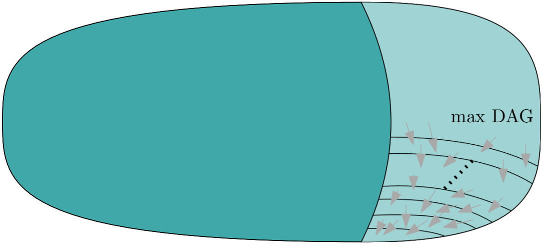

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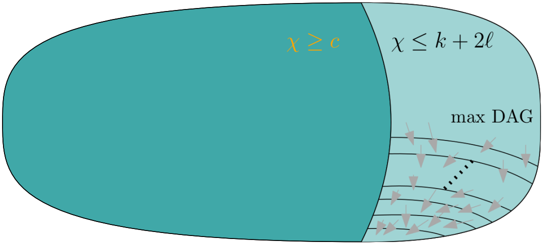

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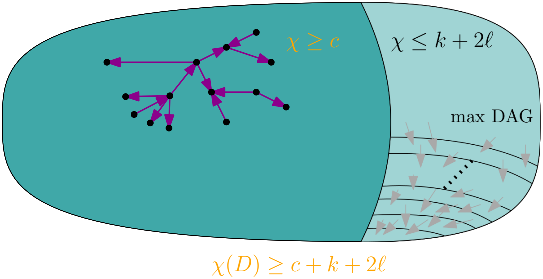

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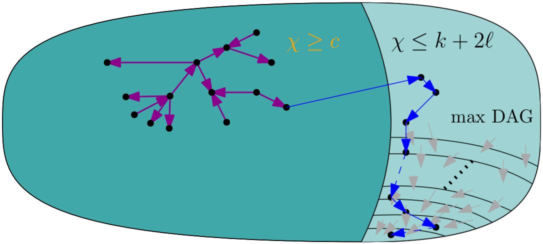

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$\chi(D) \geq 3k\sqrt{k} \implies D$ contains k-arborescences

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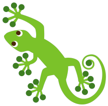
Merci!



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