Oriented trees in digraphs of high chromatic number

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joint work with
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Question
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- subgraphs: solved, trivial (next slide)
- subdigraphs: this talk!
Question

Given $H$, is there some $k$ s.t. $\chi(G) \geq k \implies G$ contains $H$ as a subgraph?
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There exist graphs of arbitrarily large girth and $\chi$. 
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**Theorem (Erdős, Hajnal ’1966)**

*There exist graphs of arbitrarily large girth and* $\chi$.

$\iff$ if $H$ is not a tree, *no.*
χ-vertex trees in simple graphs

Graph $G$: find trees on $\chi(G)$ vertices as subgraphs, tight.
\(\chi\)-vertex trees in simple graphs

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- start with any colouring
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- stack left,

![Graph diagram](Image)
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- What if we \textbf{orient} \(G\)? No guarantees on directions of arcs...
- \textit{This talk}: find \(k\)-vertex oriented subtrees when \(\chi \geq f(k)\).
Burr’s conjecture

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If a digraph $D$ satisfies $\chi(D) \geq 2k - 2$, then $D$ contains every oriented tree on $k$ vertices as a subgraph.
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If a digraph $D$ satisfies $\chi(D) \geq \frac{k^2}{2} - \frac{k}{2} + 1$, then $D$ contains every oriented tree on $k$ vertices.
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- $\geq 2k - 2 \implies$ paths, two-blocks paths*, diameter 3 trees
- $\geq 10k \implies$ antidirected trees, $\leq 4$ blocks paths
Our results

**Theorem (Bessy, Gonçalves, R. ’23+)**

*If a digraph $D$ satisfies $\chi(D) \geq 5k\sqrt{k}$, then $D$ contains every oriented tree on $k$ vertices.*
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Definition An out-arborescence is an oriented tree with:

- unique source $u$, root
- $\forall v \neq u$, one path from $u$ to $v$. 
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- Assume $\chi(D) \geq c$ $\implies$ $k$-vertex $T$,
- then $\chi(D) \geq c + 2(k + \ell) \implies T$ with $\ell$ out (/in) leaves glued to $T$. 

![Diagram showing a tree $T$ with $\ell$ leaves added to it]
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$\chi(D) \geq c + 2(k + \ell) 
\implies O(k^2)$. 

\[ \chi(D) \geq c + 2(k + \ell) \]

\[ \chi \geq c \]

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\[ \{d^+ < k + \ell\} \]
Gluing a path

- Last technique: good for lots of leaves,
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- but adding a directed path $\overrightarrow{P_\ell}$ costs $O(k\ell)$...
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max DAG
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Arborescences in digraphs

Tools for induction

- We can glue leaves or a path to $T$ by paying $O(k)$ colours
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- Do this a minimal number of times...

Observation: an out-arborescence contains either $\geq p^k$ out-leaves, or can be partitioned into $\leq p^k$ paths.

Induction

If $\leq p^k$ leaves, done: pay $3^k$ for each of the $p^k$ paths
If $\geq p^k$ out-leaves, pay $2^k$ to add them, induct...

$\chi(D) \geq 3^k p^k \implies D$ contains $k$-arborescences
Arborescences in digraphs

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Conclusion

- same gluing: \( \chi \geq (b-1)k \implies b\)-blocks paths,
- General oriented trees: same approach, harder gluing...
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Merci!
References


- Louigi Addario-Berry, Frédéric Havet, Cláudia Linhares Sales, Bruce Reed, Stéphan Thomassé, **Oriented trees in digraphs**, *Discrete Mathematics*, Volume 313, Issue 8, 2013, 967-974.