1-extendable partition of graphs

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November 21, 2023

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Context

Structural results
  Unit Disk Graphs
  Extremal properties

Cographs

Conclusion and further research
Context
Wi-Fi Network:

\( S(G) \): set of independent sets of \( G \).

\( p_v \): Probability of access of node \( v \).

\[
p_v = \frac{\sum_{S \in S(G), v \in S} \theta^{|S|}}{\sum_{S \in S(G)} \theta^{|S|}}
\]

where \( \theta >> 1 \) if a “physical parameter”\(^1\).

Wireless Networks

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Example

$$p_a = \frac{\theta^2 + \theta}{\theta^2 + 4\theta} \quad p_b = \frac{\theta}{\theta^2 + 4\theta}$$

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When $\theta \to +\infty$, 

$$p_v \sim \frac{\text{nb of max. indep. sets of } G \text{ containing } v}{\text{nb of max. indep. sets of } G} \quad := \tilde{p}_v$$
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**Example**

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Example

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BAD
What's the difference between a good and a bad network?

**Definition**
A graph $G = (V, E)$ is **1-extendable** if any vertex belongs to an MIS.

**Example**

A 1-extendable graph.

If $G = (V, E)$ is 1-extendable, for any $v \in V$, $\tilde{p}_v > 0 \rightarrow$ Minimal fairness, **Good**
What control do we have?

If the graph is not 1-extendable, what can we do?
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If the graph is not 1-extendable, what can we do? Assign a channel to each vertex.

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Example
**Definition (Berge 78)**
A graph $G$ is 1-extendable if each vertex belongs to an MIS.

**Theorem (Bergé, Busson, Feghali, Watrigant 2022)**
Testing 1-extendability is $NP$-hard, even on unit disk graph.
1-extendable and well-covered graphs

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1-Extendable Partition

**Input** : A graph $G = (V, E)$ and an integer $k$.

**Question** : Can we find a partition $V = V_1 \cup ... \cup V_k$ such that $G[V_i]$ is 1-extendable for any $1 \leq i \leq k$ ?
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**Theorem**
1-Extendable $k$-Partition is NP-hard for any fixed $k$. 
\( \chi_{1\text{-ext}}(G) \): smallest integer \( k \) such that \( G \) has a partition into \( k \) 1-extendable induced subgraphs.
Structural results
Definition
A graph $G = (V, E)$ is a unit disk graph if there exists a mapping $f : V \rightarrow \mathbb{R}^2$ such that $uv \in E$ if, and only if, $\|f(u) - f(v)\| \leq 1$.

Theorem
For any unit disk graph $G$, $\chi_{1\text{-ext}}(G) \leq 7$. 
Unit disk graphs: Model for wireless networks

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Extremal properties of $\chi_{1\text{-ext}}$

**Theorem**

*For any graph $G$ with $n$ vertices, $\chi_{1\text{-ext}}(G) \leq 2\sqrt{n}$.***
Lemma
For any graph $G$, $\chi_{1\text{-ext}}(G) \leq \alpha(G)$.

Proof.
If $\alpha(G) = 1$, then $G$ is a clique and $\chi_{1\text{-ext}}(G) = 1$. If $\alpha(G) > 1$, let $S$ be the set of vertices of $G$ that are in an MIS. Notice that:

- $G[S]$ is 1-extendable;
- $\alpha(G - S) \leq \alpha(G) - 1$.

By induction hypothesis, $\chi_{1\text{-ext}}(G - S) \leq \alpha(G) - 1$ and use one color for $S$. 

$\square$
Theorem
For any graph $G$ with $n$ vertices, $\chi_{1\text{-ext}}(G) \leq 2\sqrt{n}$.

Proof.
If $\alpha(G) > \sqrt{n}$, extract an MIS $S$, use one color for $S$ and recursively color $G - S$.

If $\alpha(G) \leq \sqrt{n}$, use $\alpha(G)$ colors with the previous lemma.
Is $O(\sqrt{n})$ optimal? Consider the following complete multipartite graph $G_n$:

Proposition $\chi_{1\text{-ext}}(G_n) = \Theta(\log(n))$. 

\[O(\sqrt{n})\]
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**Proposition**

$\chi_{1\text{-ext}}(G_n) = \Theta(\log(n))$. 

[Diagram of a complete multipartite graph $G_n$]
Cographs
A cograph is defined recursively as follows:

- A graph with a single vertex is a cograph.
- If $G_1$ and $G_2$ are both cographs, then $G_1 \cup G_2$ and $G_1 + G_2$ are cographs.
1-extendable cographs

Proposition
Given two graphs $G_1$ and $G_2$,

1. $G_1 \cup G_2$ is 1-extendable iff both $G_1$ and $G_2$ are 1-extendable;

2. $G_1 + G_2$ is 1-extendable iff both $G_1$ and $G_2$ are 1-extendable and $\alpha(G_1) = \alpha(G_2)$. 
**Theorem**
For any cograph $G$, $\chi_{1\text{-ext}}(G) \leq \log_2(\alpha(G)) + 1$.

**Idea of the proof.**
Find a partition of $V(G) = V_1 \sqcup V_2$ with:

- $G[V_1]$ 1-extendable;
- $\alpha(G[V_2]) \leq \alpha(G)/2$
Lemma

For any cograph $G = (V, E)$ and any $k \in \{0, ..., \alpha(G)\}$, there exists a partition of the vertices into two subsets $V_1$ and $V_2$ such that

- $G[V_1]$ is 1-extendable;
- $\alpha(G[V_1]) = k$;
- $\alpha(G[V_2]) \leq \max(k - 1, \alpha(G) - k)$

Proof of the theorem.
Apply the lemma for $k = \frac{\alpha(G)}{2}$, $G[V_1]$ 1-extendable and $\alpha(G[V_2]) \leq \frac{\alpha(G)}{2}$. Continue recursively with $G[V_2]$. \qed
Proof of the lemma, induction case $G = G_1 + G_2$

Let $k \in \{0, ..., \alpha(G)\}$, we apply the induction hypothesis on $(G_1, k)$ and $(G_2, k)$.

Both 1-extendable with $\alpha = k$

No large MIS
Proof of the lemma, induction case $G = G_1 \cup G_2$

Let $k \in \{0, ..., \alpha(G)\}$, we apply the induction hypothesis on $(G_1, k_1)$ and $(G_2, k_2)$, where

$$k_1 = k \frac{\alpha(G_1)}{\alpha(G_1) + \alpha(G_2)} \quad k_2 = k \frac{\alpha(G_2)}{\alpha(G_1) + \alpha(G_2)} \quad k_1 + k_2 = k$$

Both 1-extendable with

$\alpha_1 = k_1$ and $\alpha_2 = k_2$

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Case of cographs

**Theorem**

For any cograph $G$, $\chi_{1\text{-ext}}(G) \leq \log_2(\alpha(G)) + 1$, and the bound is tight.
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There exists an algorithm that solves the **1-EXTENDABLE k-PARTITION** problem on cographs in time $\mathcal{O}(n^k)$. 
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**Theorem**
There exists an algorithm that solves the $1$-$\text{Extendable } k$-$\text{Partition}$ problem on cographs in time $O(n^k)$.

**Corollary**
The $1$-$\text{Extendable Partition}$ problem can be solved in time $O(n^{\text{poly}(\log(n))})$ (quasi-polynomial), and thus is not NP-Hard on cographs, unless the ETH is false.
**Theorem**
For any cograph $G$, $\chi_{1\text{-ext}}(G) \leq \log_2(\alpha(G)) + 1$, and the bound is tight.

**Theorem**
There exists an algorithm that solves the 1-Extendable $k$-Partition problem on cographs in time $O(n^k)$.

**Corollary**
The 1-Extendable Partition problem can be solved in time $O(n^{\text{poly}(\log(n))})$ (quasi-polynomial), and thus is not NP-Hard on cographs, unless the ETH is false.
Conclusion and further research
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- New extremal results on $\chi_{1\text{-ext}}$, tight on cographs but still a gap on arbitrary graphs.
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• Quasi-polynomial algorithm for solving the partition problem on cographs. Is polynomial possible?
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• More algorithms on geometric graphs (unit disk graphs and disk graphs).
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THANKS