Reconfiguration of plane trees in convex geometric graphs

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November 23, 2023

Two **solutions** A and B of a problem P.

Reconfiguration

Two **solutions** A and B of a problem P.

Can we transform A into B?

Reconfiguration

Two **solutions** A and B of a problem P.

Can we transform A into B via a sequence of **elementary steps** while keeping solutions of P all along?

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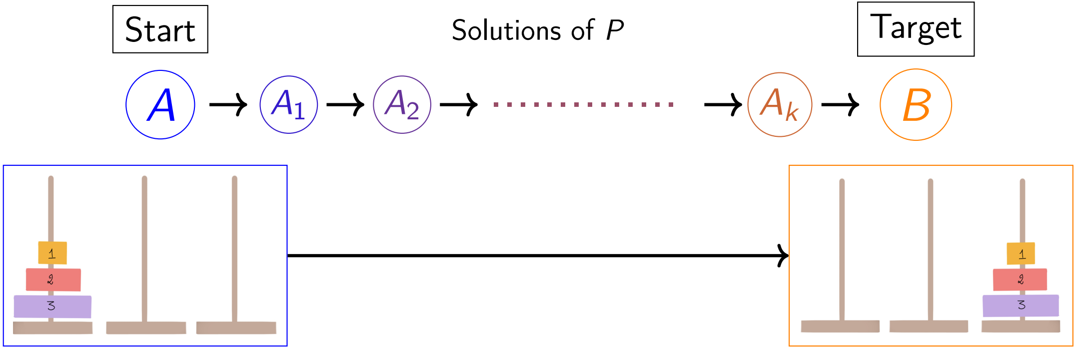

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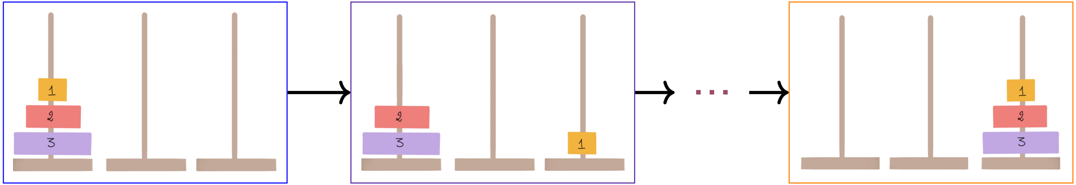
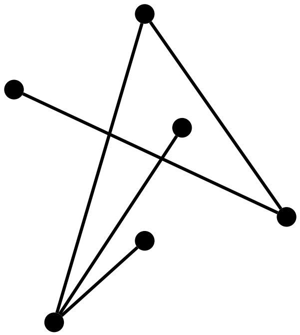


Spanning trees

- **Solutions:** Two spanning trees on a set of n points



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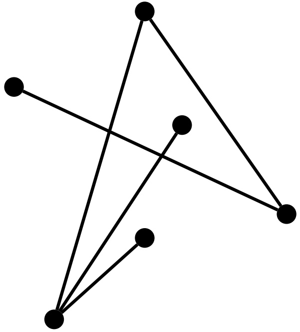
- **Elem. step:** Flip (remove an edge, then add another one)



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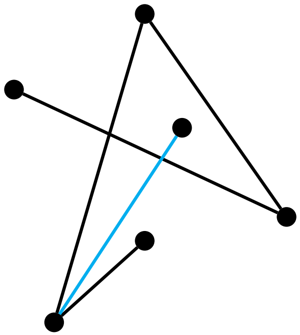

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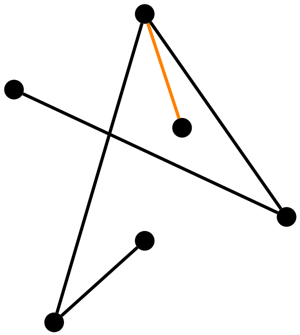

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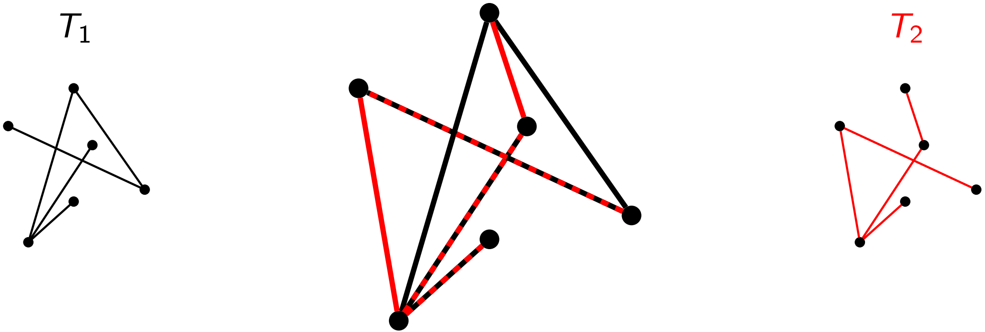

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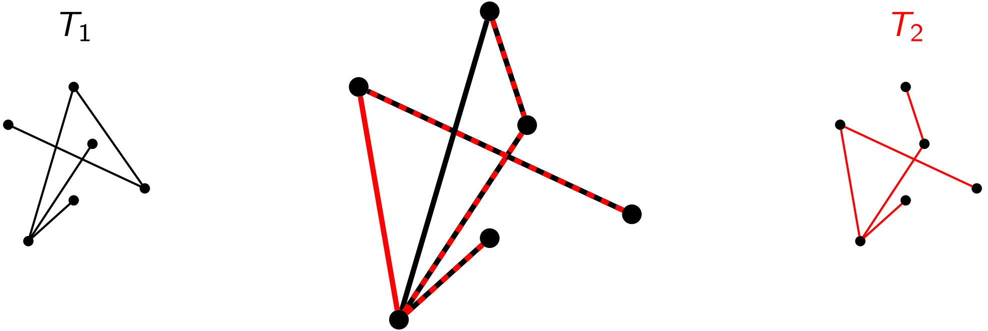

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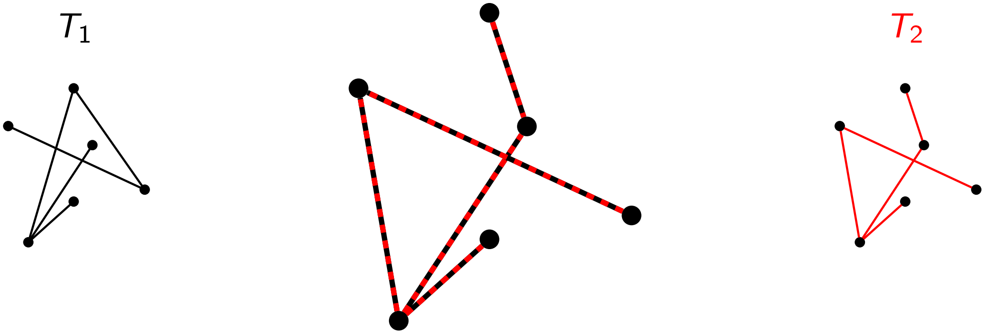

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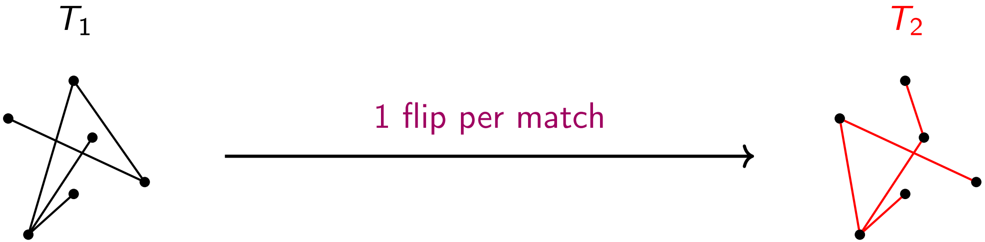

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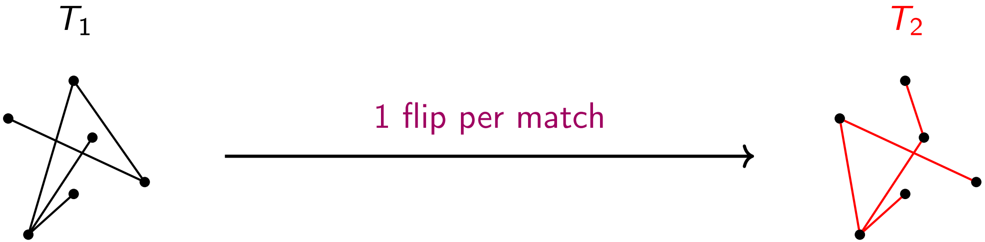

Theorem (folklore)

A minimal transformation from a spanning tree T_1 to another spanning tree T_2 uses exactly $d(T_1, T_2)$ flips.

Symmetric difference

Definitions

$$\Delta(T_1, T_2) = (T_1 \setminus T_2) \cup (T_2 \setminus T_1)$$

$$d(T_1, T_2) = |\Delta(T_1, T_2)|/2 = |T_1 \setminus T_2| = |T_2 \setminus T_1|$$

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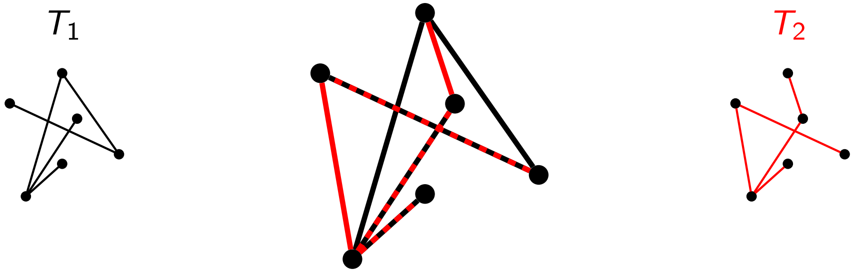

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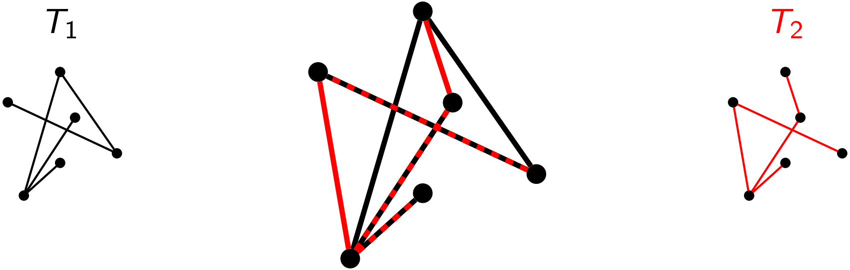

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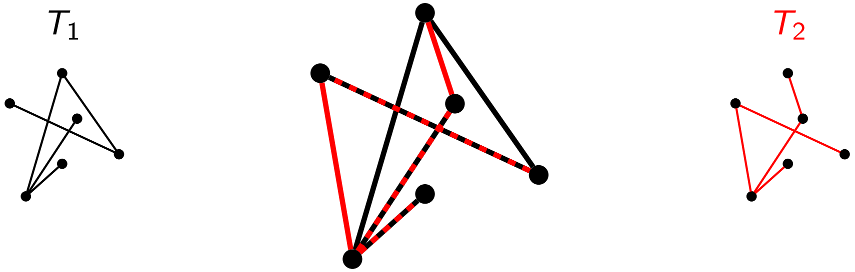
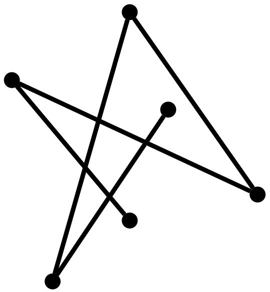


- d = half of the size of the symmetric difference.

Non-crossing spanning tree on a convex set

- **Solutions:** Two spanning trees on a set of n points



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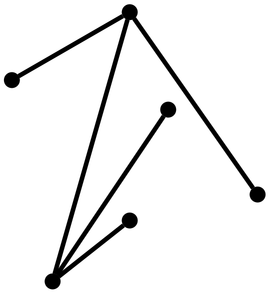
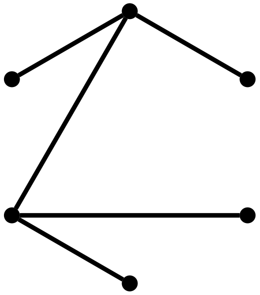


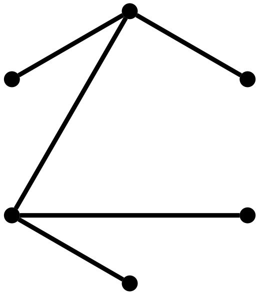
Non-crossing spanning tree on a convex set

- **Solutions:** Two non-crossing spanning trees on a set of n points in convex position



Non-crossing spanning tree on a convex set

- **Solutions:** Two non-crossing spanning trees on a set of n points in convex position
- **Elem. step:** Flip



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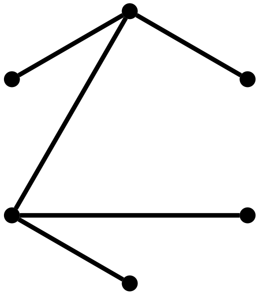
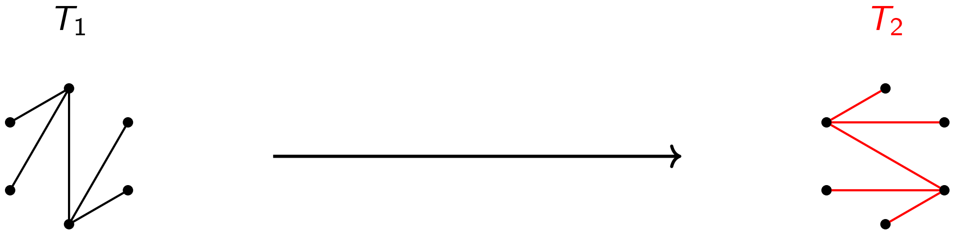


Tree = non-crossing spanning tree on a convex set.

Reconfiguration of n.-c. spanning trees on convex set



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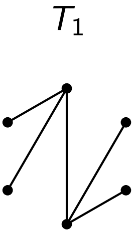
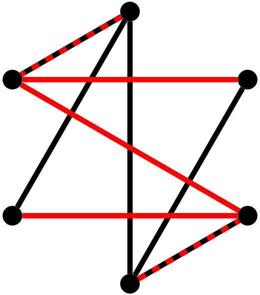
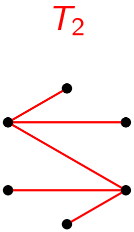




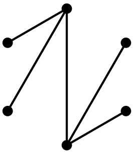


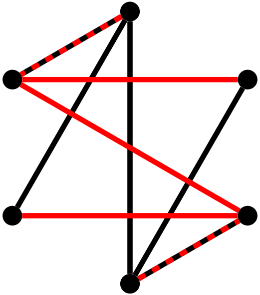
Reconfiguration of n.-c. spanning trees on convex set

Avis and Fukuda ('96)

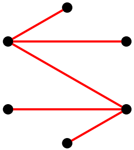
For every pair of non-crossing spanning trees T_1 and T_2, there exists a transformation from T_1 to T_2 using flips.

T_1





T_2



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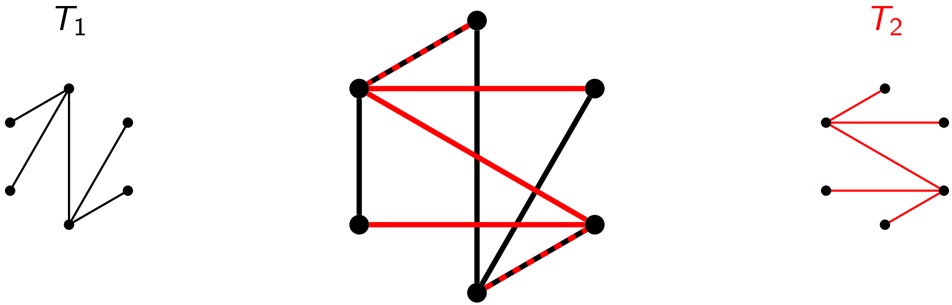

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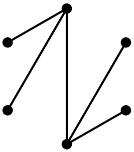

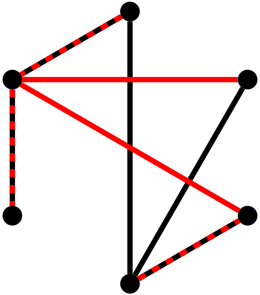

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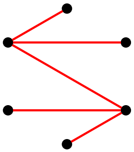

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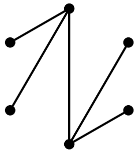

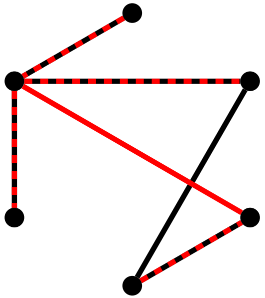

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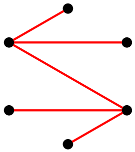

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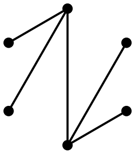

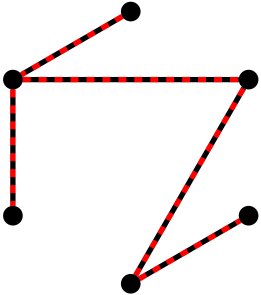

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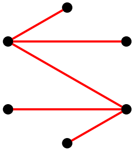

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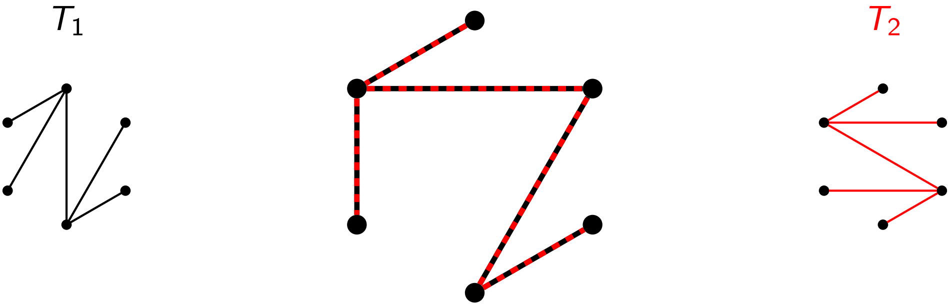
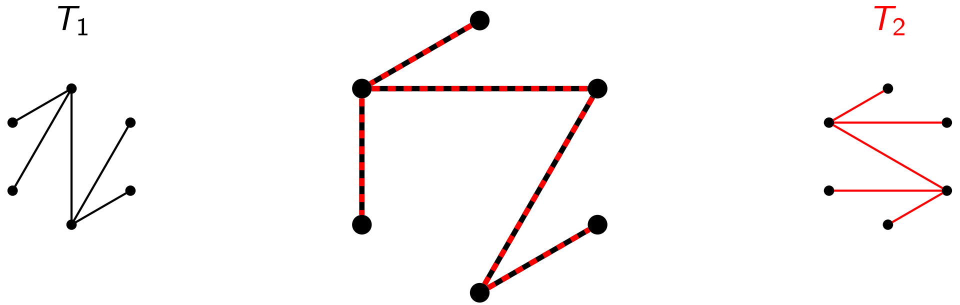


How many flips are needed in the worst case?

Avis and Fukuda ('96)

For every pair of non-crossing spanning trees T_1 and T_2, there exists a transformation from T_1 to T_2 using at most $2n - 4$ flips.



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Existing results

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- Upper Bound:
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  - $\frac{3}{2}n - 5$ flips (Hernando et al., 1999).

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Conjecture

For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $\frac{3}{2}n$ flips.

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  - $\approx 1.95d$ flips.
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  - $\frac{5}{3}d$ flips.

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For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $\frac{3}{2}n$ flips.

Result on upper bound

Theorem (Bousquet, dM, Pierron, Wesolek)

For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $c \cdot d$ flips with:

$$c = \frac{1}{12}(22 + \sqrt{2}) \approx 1.95$$

Result on upper bound

Theorem (Bousquet, dM, Pierron, Wesolek)

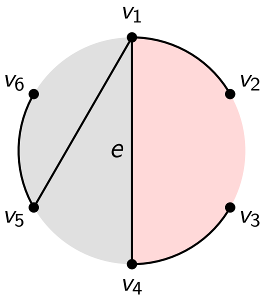
For every pair of trees T_1 and T_2, there is a transformation from T_1 to T_2 using at most $c \cdot d$ flips with:

$$c = \frac{1}{12}(22 + \sqrt{2}) \approx 1.95$$

$\implies$ there is always a transformation using at most $c \cdot n$ flips.

Very good side

Side (of a non-border edge) : one subset of the convex set when cutting along the edge, endpoints included.



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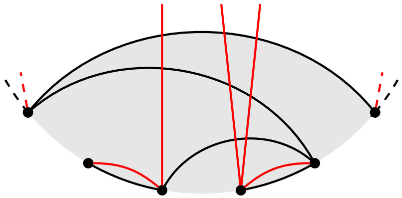

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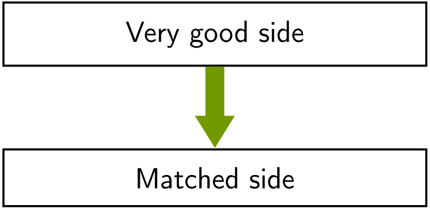

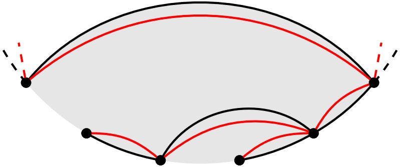

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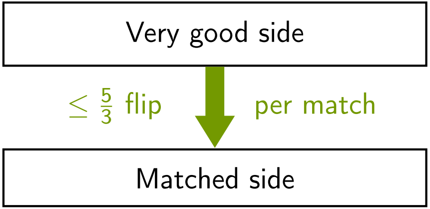
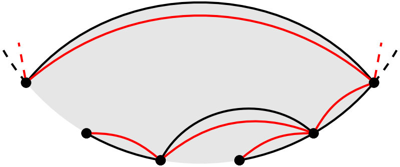
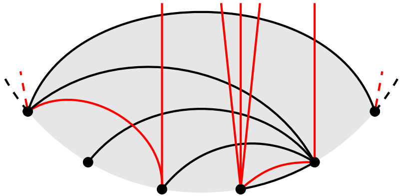




Proof sketch

Existing side with
weaker properties

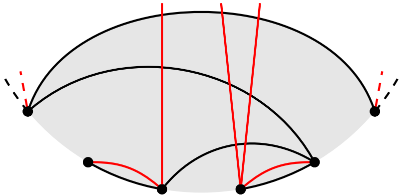


Proof sketch

Existing side with weaker properties

↓

Very good side



Proof sketch

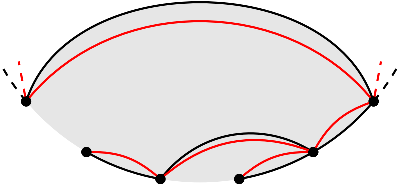
Existing side with weaker properties

↓

Very good side

↓ $\leq \frac{5}{3}$ flips per match

Matched side



Proof sketch

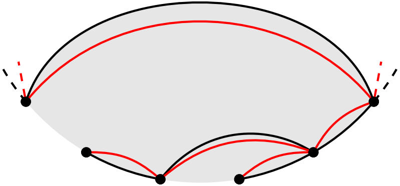
Existing side with weaker properties

≤ 2 flips per match

Very good side

$\leq \frac{5}{3}$ flips per match

Matched side

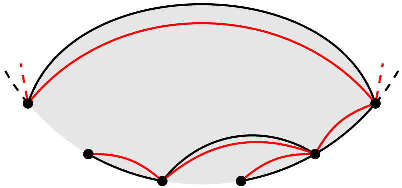


End of proof

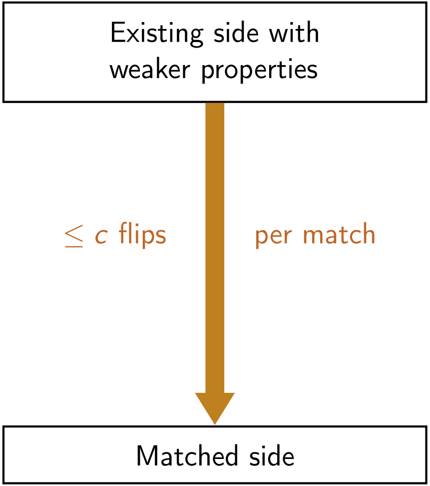
Existing side with weaker properties

$\leq c$ flips per match

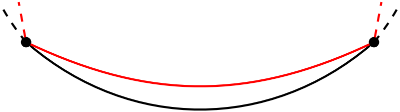
Matched side



End of proof



Existing side with weaker properties
$\leq c$ flips
per match
Matched side



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Other models of elementary steps:

- Non-crossing flips: $2d \leq$ number of n-c flips $\leq 2d$
- Rotations: $\frac{7}{3}d \leq$ number of rotations $\leq 3d$

Thanks for your attention

END

Result 2 : lower bound

$\frac{3}{2}n - 5$ (Hernando et al.).

Theorem (Bousquet, dM, Pierron, Wesolek)

For every $k = 0 \bmod 3$, there exists a pair of trees T_k, T'_k such that $d(T_k, T'_k) = k$ and whose minimal transformation contains exactly $\frac{5}{3}k$ flips.