Asymptotiques pour les séries graphiquement divergentes

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Motivation: connected undirected graphs

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$$p_n = 1 - \binom{n}{1}\frac{2}{2^n} - 2\binom{n}{3}\frac{2^6}{2^{3n}} - 24\binom{n}{4}\frac{2^{10}}{2^{4n}} + O\left(\frac{n^5}{2^{5n}}\right)$$

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Can we see the structure? What is the interpretation?

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4. Monteil, N., 2021:
$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $\mathrm{it}_k = \#\{$irreducible labeled tournaments of size $k\}$.

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Wright, 1971: $$r_n = \sum_{k=0}^{r-1} \frac{\omega_k(n)}{2^{kn}} \cdot \frac{n!}{(n + [k/2] - k)!} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

$$\omega_k(n) = \sum_{\nu=0}^{[k/2]} \gamma_\nu \xi_{k-2\nu} \frac{2^{k(k+1)/2}}{2^{\nu(k-\nu)}} (n + [k/2] - k) \ldots (n + \nu + 1 - k),$$

$$\gamma_0 = 1,\ \gamma_\nu = \sum_{s=0}^{\nu-1} \frac{\gamma_s \eta_{\nu-s}}{(\nu-s)!},\quad \sum_{\nu=0}^{\infty} \xi_\nu z^\nu = \left(1 - \sum_{s=0}^{\infty} \frac{\eta_s}{2^{s(s-1)/2}} \frac{z^s}{s!}\right)^2,$$

$$\eta_1 = 1,\quad \eta_s = 2^{s(s-1)} - \sum_{t=1}^{s-1} \binom{s}{t} 2^{(s-1)(s-t)} \eta_t.$$

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Motivation: strongly connected directed graphs

Summary. The probability r_n has an expansion

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{mn}} \sum_{\ell=0}^{L_m} n^{\underline{\ell}} \cdot a^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $n^{\underline{\ell}} = n(n-1)\dots(n-\ell+1)$ are falling factorials.

Observation. The array of coefficients $(a^{\circ}_{m,\ell})_{m,\ell=0}^{\infty}$ can be assembled into a bivariate generation function.

Question. Can we express this bivariate generating function explicitly in terms of other known generating functions?

Factorially divergent series (Borinsky)

$$a_n = \alpha^{n+\beta}\Gamma(n+\beta)\left(c_0 + \frac{c_1}{\alpha(n+\beta-1)} + \frac{c_2}{\alpha^2(n+\beta-1)(n+\beta-2)} + \dots\right)$$

$$\sum_{n=0}^{\infty} a_n z^n \xmapsto{\mathcal{A}_\beta^\alpha} \sum_{n=0}^{\infty} c_n z^n$$

Properties:

- $\big(\mathcal{A}_\beta^\alpha(A \cdot B)\big)(z) = A(z) \cdot (\mathcal{A}_\beta^\alpha B)(z) + B(z) \cdot (\mathcal{A}_\beta^\alpha A)(z)$
- $\big(\mathcal{A}_\beta^\alpha(A \circ B)\big)(z) = A'\big(B(z)\big) \cdot (\mathcal{A}_\beta^\alpha B)(z)$
$$+ \left(\frac{z}{B(z)}\right)^\beta \exp\left(\frac{1}{\alpha}\left(\frac{1}{z} - \frac{1}{B(z)}\right)\right)(\mathcal{A}_\beta^\alpha A)\big(B(z)\big)$$

Graphically divergent series

$$a_n = \alpha^{\beta\binom{n}{2}} \left[\sum_{m=0}^{r-1} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^{\ell} \, a^{\circ}_{m,\ell} + O\left(\frac{n^{L_r}}{\alpha^{rn}} \right) \right]$$

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,

Graphically divergent series

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$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad \xrightarrow{\mathcal{Q}_{\alpha}^{\beta}} \quad \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^{\ell}$$

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$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \xrightarrow{\ \mathcal{Q}_\alpha^\beta\ } \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^{\ell}$$

graphically divergent series

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,
- $\mathfrak{G}_\alpha^\beta$ is the set of graphically divergent series,

Graphically divergent series

$$a_n = \alpha^{\beta\binom{n}{2}} \left[\sum_{m=0}^{r-1} \frac{1}{\alpha^{mn}} \sum_{\ell=0}^{L_m} n^{\ell}\, a^{\circ}_{m,\ell} + O\left(\frac{n^{L_r}}{\alpha^{rn}}\right) \right]$$

$$\boxed{\mathcal{Q}_\alpha^\beta \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{C}_\alpha^\beta}$$

coefficient generating function

of type (α, β)

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \quad \overset{\mathcal{Q}_\alpha^\beta}{\longmapsto} \quad \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m}{\alpha^{\frac{1}{\beta}\binom{m}{2}}} w^{\ell}$$

graphically divergent series

- $\alpha \in \mathbb{R}_{>1}$ and $\beta \in \mathbb{Z}_{>0}$ are parameters,
- $\mathfrak{G}_\alpha^\beta$ is the set of graphically divergent series,
- $\mathfrak{C}_\alpha^\beta$ is the set of bivariate power series.

Properties, part I

1. The set $\mathfrak{G}_\alpha^\beta$ forms a ring with

$$\big(\mathcal{Q}_\alpha^\beta(A+B)\big)(z,w) = (\mathcal{Q}_\alpha^\beta A)(z,w) + (\mathcal{Q}_\alpha^\beta B)(z,w)$$

and

$$\begin{aligned}\big(\mathcal{Q}_\alpha^\beta(A\cdot B)\big)(z,w) &= A\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta B)(z,w)\\ &\quad + B\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big)\cdot(\mathcal{Q}_\alpha^\beta A)(z,w)\,.\end{aligned}$$

2. Derivation:

$$\big(\mathcal{Q}_\alpha^\beta A'\big)(z,w) = \alpha^{-\frac{\beta+1}{2}} z^{-\beta}\left((\mathcal{Q}_\alpha^\beta A)(z,w) + \frac{\partial}{\partial w}(\mathcal{Q}_\alpha^\beta A)(z,w)\right).$$

Properties, part II

3. Composition (interpretation of Bender's theorem): if
   - F is analytic in a neighbourhood of the origin,
   - $a_0 = 0$,
   - $H(z) = \frac{\partial}{\partial x} F(x) \Big|_{x=A(z)}$,

   then $\quad F \circ A \in \mathfrak{G}_\alpha^\beta \quad$ and

$$\big(\mathcal{Q}_\alpha^\beta (F \circ A)\big)(z, w) = H\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z, w).$$

4. Powers: if $\quad m \in \mathbb{Z}_{\geqslant 0} \quad$ (or $m \in \mathbb{Z}$ and $a_0 = 1$), $\quad$ then

$$\big(\mathcal{Q}_\alpha^\beta A^m\big)(z, w) = m \cdot A^{m-1}\big(\alpha^{\frac{\beta+1}{2}} z^\beta w\big) \cdot (\mathcal{Q}_\alpha^\beta A)(z, w)\,.$$

Connected graphs

Monteil, N., 2021: For every $r \geqslant 1$, the probability p_n that a random graph of size n is connected satisfies

$$p_n = 1 - \sum_{k=1}^{r-1} \mathrm{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $\mathrm{it}_k = \#\{$irreducible labeled tournaments of size $k\}$.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ *of connected graphs satisfies*

$$(\mathcal{Q}_2^1\,\mathrm{CG})(z,w) = \frac{1}{\mathrm{G}(2zw)} = 1 - \mathrm{IT}(2zw).$$

Key ideas: $(\mathcal{Q}_2^1\,\mathrm{G})(z,w) = 1,\ \mathrm{CG}(z) = \log\left(\mathrm{G}(z)\right),\ \frac{1}{\mathrm{G}(z)} = \frac{1}{\mathrm{T}(z)} = 1 - \mathrm{IT}(z).$

Irreducible tournaments

Monteil, N., 2021: For every $r \geqslant 1$, the probability q_n that a random tournament of size n is irreducible satisfies

$$q_n = 1 - \sum_{k=1}^{r-1} \left(2\mathrm{it}_k - \mathrm{it}_k^{(2)}\right) \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{nk}} + O\left(\frac{n^r}{2^{rn}}\right),$$

where $\mathrm{it}_k^{(2)} = \#\{$tournaments with two irreducible parts of size $k\}$.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ of irreducible tournaments satisfies

$$(\mathcal{Q}_2^1\,\mathsf{IT})(z,w) = \big(1 - \mathsf{IT}(2zw)\big)^2.$$

Key ideas: $(\mathcal{Q}_2^1\,\mathsf{T})(z,w) = 1$, $\mathsf{IT}(z) = 1 - \dfrac{1}{\mathsf{T}(z)}$, $\dfrac{1}{\mathsf{T}^2(z)} = \big(1 - \mathsf{IT}(z)\big)^2$.

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\mathrm{SCD}(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right) .$$

- Exponential Hadamard product:

$$\left(\sum_{n=0} a_n \frac{z^n}{n!}\right) \odot \left(\sum_{n=0} b_n \frac{z^n}{n!}\right) = \left(\sum_{n=0} a_n b_n \frac{z^n}{n!}\right) .$$

- Exponential Hadamard product (with $G(z)$) changes:
  - the rate of growth,
  - the type of coefficient generating function.

Transitions, part I

Theorem (Dovgal, de Panafieu, 2019)

The exponential generating function of strongly connected digraphs satisfies

$$\mathrm{SCD}(z) = -\log\left(G(z) \odot \frac{1}{G(z)}\right).$$

- If $\beta > 1$, then

$$\Delta_\alpha \colon \mathfrak{G}_\alpha^\beta \to \mathfrak{G}_\alpha^{\beta-1}$$

  is defined by

$$\Delta_\alpha\left(\sum_{n=0}^{\infty} f_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{f_n}{\alpha^{\binom{n}{2}}} \frac{z^n}{n!}.$$

- $F(z) \odot G(z) = \Delta_2^{-1} F(z)\,.$

Transitions, part II

- If $\alpha \in \mathbb{R}_{>1}$ and $\beta_1, \beta_2 \in \mathbb{Z}_{>0}$, then

$$\Phi_\alpha^{\beta_1,\beta_2} : \mathfrak{C}_\alpha^{\beta_1} \to \mathfrak{C}_\alpha^{\beta_2}$$

  is defined as

$$\sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m w^\ell}{\alpha^{\frac{1}{\beta_1}\binom{m}{2}}} \xmapsto{\Phi_\alpha^{\beta_1,\beta_2}} \sum_{m=0}^{\infty} \sum_{\ell=0}^{L_m} a^{\circ}_{m,\ell} \frac{z^m w^\ell}{\alpha^{\frac{1}{\beta_2}\binom{m}{2}}} .$$

- The following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{G}_\alpha^{\beta_1} & \xrightarrow{\mathcal{Q}_\alpha^{\beta_1}} & \mathfrak{C}_\alpha^{\beta_1} \\ {\scriptstyle \Delta_\alpha^{\beta_1-\beta_2}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi_\alpha^{\beta_1,\beta_2}} \\ \mathfrak{G}_\alpha^{\beta_2} & \xrightarrow{\mathcal{Q}_\alpha^{\beta_2}} & \mathfrak{C}_\alpha^{\beta_2} \end{array}$$

Strongly connected directed graphs, part I

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ *of strongly connected digraphs satisfies*

$$(\mathcal{Q}_2^2\,\mathrm{SCD})(z,w) = \mathrm{SSD}(2^{3/2}z^2 w)\cdot \Phi_2^{1,2}\big(1-\mathrm{IT}(2zw)\big)^2 .$$

where $\mathrm{SSD}(z)$ *is the exponential generating function of semi-strong digraphs.*

Key ideas (Dovgal, de Panafieu, 2019; Monteil, N., 2021):

- $\mathrm{SCD}(z) = -\log\left(G(z)\odot\dfrac{1}{G(z)}\right) = -\log\left(1-\Delta_2^{-1}\mathrm{IT}(z)\right),$
- $\mathrm{SSD}(z) = \left(G(z)\odot\dfrac{1}{G(z)}\right)^{-1} = \dfrac{1}{1-\Delta_2^{-1}\mathrm{IT}(z)}.$

Strongly connected directed graphs, part II

Corollary

For every $r \geqslant 1$, the probability r_n that a random labeled digraph of size n is strongly connected satisfies

$$r_n = \sum_{m=0}^{r-1} \frac{1}{2^{nm}} \sum_{\ell=\lceil m/2 \rceil}^{m} n^{\ell} \mathfrak{scd}^{\circ}_{m,\ell} + O\left(\frac{n^r}{2^{rn}}\right),$$

where

- $\mathfrak{scd}^{\circ}_{m,\ell} = \dfrac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \dfrac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \dfrac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!}$,
- $\mathfrak{ssd}_k$ *is the number of semi-strong digraphs of size* k,
- $\mathfrak{it}_k$ *is the number of irreducible tournaments of size* k,
- $\mathfrak{it}^{(2)}_k$ *is the number of tournaments of size* k *with two irreducible components.*

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where

- $$\mathfrak{scd}^{\circ}_{m,\ell} = \frac{2^{m(m+1)/2}}{2^{\ell(m-\ell)}} \frac{\mathfrak{ssd}_{m-\ell}}{(m-\ell)!} \frac{\mathbb{I}_{m=2\ell} - 2\mathfrak{it}_{2\ell-m} + \mathfrak{it}^{(2)}_{2\ell-m}}{(2\ell-m)!},$$
- *Interpretation of Wright's coefficients:*

$$\eta_k = 2^{\binom{k}{2}} \mathfrak{it}_k, \qquad \gamma_k = \frac{\mathfrak{ssd}_k}{k!}, \qquad \xi_k = \frac{\mathbb{I}_{k=0} - 2\mathfrak{it}_k + \mathfrak{it}^{(2)}_k}{k!}.$$

Conclusion

1. We have constructed a tool for manipulating coefficients of asymptotic expansions.

2. Transfers extend to graphic families with marked patterns:
   - connected components in graphs, strongly connected components in digraphs, contradictory components in 2-sat,
   - source-like, sink-like, isolated components,
   - any graphically divergent series with marking variables.

3. Bonus: combinatorial explanations of the expansion coefficients.

Thank you for your attention!

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Fixed number of connected components in a graph

Observation: $\mathrm{G}(z;t) = \exp\big(t \cdot \mathrm{CG}(z)\big)$,
where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ of graphs with the marking variable t satisfies

$$(\mathcal{Q}_2^1\, G)(z,w;t) = t \cdot \mathrm{G}(2zw; t-1) = t \cdot \mathrm{G}(2zw;t) \cdot \big(1 - \mathrm{IT}(2zw)\big).$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1\, G)(z,w;t) = \frac{\mathrm{CG}^m(2zw)}{m!} \cdot \big(1 - \mathrm{IT}(2zw)\big)$$

is the coefficient generating function for graphs with $(m+1)$ connected components, $m \in \mathbb{Z}_{\geqslant 0}$.

Fixed number of irreducible parts in a tournament

Observation: $\mathsf{T}(z;t) = \dfrac{1}{1 - t \cdot \mathsf{IT}(z)}$,

where t marks the number of irreducible parts.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,1)$ of tournaments with the marking variable t satisfies

$$(\mathcal{Q}_2^1\, T)(z,w;t) = t \cdot \Big(\mathsf{T}(2zw;t) \cdot \big(1 - \mathsf{IT}(2zw)\big)\Big)^2.$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1\, T)(z,w;t) = (m+1) \cdot \mathsf{IT}^m(2zw) \cdot \big(1 - \mathsf{IT}(2zw)\big)^2$$

is the coefficient generating function for tournaments with $(m+1)$ irreducible parts, $m \in \mathbb{Z}_{\geqslant 0}$.

The Erdős-Rényi model $G(n, p)$, part I

Fix $\quad p \in (0,1), \quad q = 1 - p, \quad \rho = p/q.$

Consider a random labeled graph G:

- p is the probability of edge presence;
- $q = 1 - p$ is the probability of edge absence.

$$\mathbb{P}(G) = p^{|E(G)|} q^{\binom{n}{2} - |E(G)|} = \frac{\rho^{|E(G)|}}{(\rho+1)^{\binom{n}{2}}}$$

Denote:

- $\alpha = \rho + 1 = q^{-1}$.

Then

$$G(z) = \sum_{n=0}^{\infty} (\rho+1)^{\binom{n}{2}} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \alpha^{\binom{n}{2}} \frac{z^n}{n!}.$$

The Erdős-Rényi model $G(n, p)$, part II

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2, 1)$ of connected graphs in the Erdös-Rényi model satisfies

$$(\mathcal{Q}_2^1 \,\mathrm{CG})(z, w) = \frac{1}{\mathrm{G}(2zw)} = \exp\big(-\mathrm{CG}(2zw)\big).$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2, 1)$ of graphs in the Erdös-Rényi model with the marking variable t for the number of strongly connected components satisfies

$$(\mathcal{Q}_2^1 \, G)(z, w; t) = t \cdot \mathrm{G}(2zw; t-1)$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^1 \, G)(z, w; t) = \frac{\mathrm{CG}^m(2zw)}{m!} \cdot \exp\big(-\mathrm{CG}(2zw)\big).$$

Fixed number of strongly connected components, part I

Observation: $\mathrm{SSD}(z;t) = \exp\big(t \cdot \mathrm{SCD}(z)\big)$,
where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ of semi-strong digraphs with the marking variable t satisfies

$$(\mathcal{Q}_2^2\, SSD)(z,w;t) = t \cdot \mathrm{SSD}(2^{3/2} z^2 w; t+1) \cdot \Phi_2^{1,2}\big(1 - \mathrm{IT}(2zw)\big)^2.$$

In particular,

$$[t^{m+1}](\mathcal{Q}_2^2\, SSD)(z,w;t) = \frac{\mathrm{SCD}^m(2^{3/2} z^2 w)}{m!} \cdot (\mathcal{Q}_2^2\, SCD)(z,w)$$

is the coefficient generating function for semi-strong digraphs with $(m+1)$ strongly connected components, $m \in \mathbb{Z}_{\geqslant 0}$.

Fixed number of strongly connected components, part II

Observation (Robinson, 1973):

$$\mathrm{D}(z;t) = \Delta_2^{-1}\left(\frac{1}{\Delta_2\, e^{-t\cdot \mathrm{SCD}(z)}}\right) = \Delta_2^{-1}\left(\frac{1}{\Delta_2\, \mathrm{SSD}(z;-t)}\right),$$

where t marks the number of connected components.

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,2)$ *of digraphs with the marking variable* t *satisfies*

$$(\mathcal{Q}_2^2\, D)(z,w;t) = -\Phi_2^{1,2}\left(\frac{\Phi_2^{2,1}\big((\mathcal{Q}_2^2\, SSD)(z,w;-t)\big)}{\big(\Delta_2\, SSD(2zw;-t)\big)^2}\right).$$

Asymptotics of 2-SAT formulae

Implication generating function of 2-SAT formulae:

$$\ddot{\mathrm{SAT}}(z) = \sum_{n=0}^{\infty} \mathfrak{sat}_n \frac{z^n}{2^{n^2} n!}\,.$$

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\ddot{\mathrm{SAT}}(z) = G(z) \cdot \Delta_2^2 \left(G(z) \odot \frac{1}{G(z)} \right)^{1/2}.$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2, 1)$ *of 2-SAT formulae satisfies*

$$(\mathcal{Q}_2^1 \ddot{\mathrm{SAT}})(z, w) = \frac{\ddot{\mathrm{SAT}}(2zw)}{\mathrm{G}(2zw)} = \ddot{\mathrm{SAT}}(2zw)(1 - \mathrm{IT}(2zw))\,.$$

Asymptotics of contradictory components

Observation (Dovgal, de Panafieu, Ravelomanana, 2023):

$$\mathrm{CSC}(z) = \frac{1}{2}\mathrm{SCD}(2z) + \log\left(D(z) \odot \frac{D(z)}{G(2z)} \right) .$$

Theorem (Dovgal, N., 2023+)

The coefficient generating function of type $(2,4)$ of contradictory strongly connected implication digraphs satisfy

$$(\mathcal{Q}_2^4\,\mathrm{CSC})(z,w) = \exp\left(\frac{1}{2}\mathrm{SCD}(2^{7/2}z^4w) - \mathrm{CSC}(2^{5/2}z^4w) \right) \cdot \Phi_2^{2,4}(1 - \mathrm{IT}(2^{5/2}z^2zw)).$$