# Elementary first-order model checking for sparse graphs

Jakub Gajarský
Michał Pilipczuk
Marek Sokołowski
*Giannos Stamoulis*
Szymon Toruńczyk

Institute of Informatics, University of Warsaw, Poland

22.11.2023
Journées Graphes et Algorithmes 2023

## Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: $x = y$, $adj(x, y)$
Logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$.
Quantifiers: $\exists x\ \varphi$, $\forall x\ \varphi$

# Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: $x = y$, $adj(x, y)$
Logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$.
Quantifiers: $\exists x\ \varphi$, $\forall x\ \varphi$

*"$P_3$ is an induced subgraph of $G$"*:

$$\exists x \exists y \exists z\ \Big( adj(x, y) \wedge adj(y, z) \wedge \neg adj(x, z) \Big)$$

*"$G$ has a dominating set of size 3"*:

$$\exists x_1 \exists x_2 \exists x_3\ \forall y\ \bigvee_{i \in \{1,2,3\}} \Big( y = x_i \vee adj(x_i, y) \Big)$$

## Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: $x = y$, $adj(x, y)$
Logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$.
Quantifiers: $\exists x\ \varphi$, $\forall x\ \varphi$

FO Model Checking
Input: a first-order formula $\varphi$ and a graph $G$.
Question: $G$ satisfies $\varphi$?

*"$P_3$ is an induced subgraph of $G$"*:

$$\exists x \exists y \exists z \left( adj(x, y) \wedge adj(y, z) \wedge \neg adj(x, z) \right)$$

*"$G$ has a dominating set of size 3"*:

$$\exists x_1 \exists x_2 \exists x_3\ \forall y\ \bigvee_{i \in \{1,2,3\}} \left( y = x_i \vee adj(x_i, y) \right)$$

# Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: $x = y$, $adj(x, y)$
Logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$.
Quantifiers: $\exists x\ \varphi$, $\forall x\ \varphi$

"*$P_3$ is an induced subgraph of $G$*":

$$\exists x \exists y \exists z \left( adj(x, y) \wedge adj(y, z) \wedge \neg adj(x, z) \right)$$

"*$G$ has a dominating set of size 3*":

$$\exists x_1 \exists x_2 \exists x_3\ \forall y\ \bigvee_{i \in \{1,2,3\}} \left( y = x_i \vee adj(x_i, y) \right)$$

FO Model Checking
Input: a first-order formula $\varphi$ and a graph $G$.
Question: $G$ satisfies $\varphi$?

- On general graphs, the problem is AW[*]-hard.

## Model checking first-order formulas (on graphs)

first-order logic (**FO**):

Atomic formulas: $x = y$, $adj(x, y)$
Logical connectives: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$.
Quantifiers: $\exists x\ \varphi$, $\forall x\ \varphi$

*"$P_3$ is an induced subgraph of $G$"*:

$$\exists x \exists y \exists z \left( adj(x, y) \wedge adj(y, z) \wedge \neg adj(x, z) \right)$$

*"$G$ has a dominating set of size 3"*:

$$\exists x_1 \exists x_2 \exists x_3\ \forall y\ \bigvee_{i \in \{1,2,3\}} \left( y = x_i \vee adj(x_i, y) \right)$$

FO MODEL CHECKING
Input: a first-order formula $\varphi$ and a graph $G$.
Question: $G$ satisfies $\varphi$?

- On general graphs, the problem is AW[*]-hard.
- When is it **FPT**? i.e., solvable in time $f(|\varphi|, \mathcal{C}) \cdot |G|^c$, for some function $f$ and $c \geq 1$.

## The three components of the model checking question

**FO** model checking is **FPT** on $\mathcal{C}$.

## The three components of the model checking question

[Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, & Toruńczyk, 2023]
[Dreier, Mählmann, & Siebertz, 2023]
[Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, & Toruńczyk, 2022]
[Bonnet, Kim, Thomassé, & Watrigant, 2022]
[Hliněný, Pokrývka, & Roy, 2019]
[Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Pilipczuk, Siebertz, & Toruńczyk, 2018]
[Grohe, Kreutzer, & Siebertz, 2017]
[Eickmeyer & Kawarabayashi, 2017]
[Gajarský, Hliněný, Lokshtanov, Obdržálek, & Ramanujan, 2016]
[Dvořák, Král̆, & Thomas, 2011]
[Dawar, Grohe, & Kreutzer, 2007]
[Flum & Grohe, 2001]
[Frick & Grohe, 2001]
[Seese, 1996]

**FO** model checking is **FPT** on $\mathcal{C}$.

How general $\mathcal{C}$ can be?

## The three components of the model checking question

[Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, & Toruńczyk, 2023]
[Dreier, Mählmann, & Siebertz, 2023]
[Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, & Toruńczyk, 2022]
[Bonnet, Kim, Thomassé, & Watrigant, 2022]
[Hliněný, Pokrývka, & Roy, 2019]
[Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Pilipczuk, Siebertz, & Toruńczyk, 2018]
[Grohe, Kreutzer, & Siebertz, 2017]
[Eickmeyer & Kawarabayashi, 2017]
[Gajarský, Hliněný, Lokshtanov, Obdržálek, & Ramanujan, 2016]
[Dvořák, Král̆, & Thomas, 2011]
[Dawar, Grohe, & Kreutzer, 2007]
[Flum & Grohe, 2001]
[Frick & Grohe, 2001]
[Seese, 1996]

Extensions of **FO** ?

**FO** model checking is **FPT** on $\mathcal{C}$.

How general $\mathcal{C}$ can be?

[Schirrmacher, Siebertz, Stamoulis, Thilikos, & Vigny, 2023]
[Golovach, Stamoulis, & Thilikos, 2023]
[Fomin, Golovach, Sau, Stamoulis, & Thilikos, 2023]
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny, 2022]
[Schirrmacher, Siebertz, & Vigny, 2022]
[Nešetřil, Ossona de Mendez, & Siebertz, 2022]
[Grange, 2021]
[Berkholz, Keppeler, & Schweikardt, 2018]
[Grohe & Schweikardt, 2018]
[van den Heuvel, Kreutzer, Pilipczuk, Quiroz, Rabinovich, & Siebertz, 2017]

## The three components of the model checking question

[Dreier, Eleftheriadis, Mählmann, McCarty, Pilipczuk, & Toruńczyk, 2023]
[Dreier, Mählmann, & Siebertz, 2023]
[Bonnet, Dreier, Gajarský, Kreutzer, Mählmann, Simon, & Toruńczyk, 2022]
[Bonnet, Kim, Thomassé, & Watrigant, 2022]
[Hliněný, Pokrývka, & Roy, 2019]
[Gajarský, Kreutzer, Nešetřil, Ossona de Mendez, Pilipczuk, Siebertz, & Toruńczyk, 2018]
[Grohe, Kreutzer, & Siebertz, 2017]
[Eickmeyer & Kawarabayashi, 2017]
[Gajarský, Hliněný, Lokshtanov, Obdržálek, & Ramanujan, 2016]
[Dvořák, Kráľ, & Thomas, 2011]
[Dawar, Grohe, & Kreutzer, 2007]
[Flum & Grohe, 2001]
[Frick & Grohe, 2001]
[Seese, 1996]

Extensions of **FO** ?

**FO** model checking is **FPT** on $\mathcal{C}$.

How general $\mathcal{C}$ can be?

[Schirrmacher, Siebertz, Stamoulis, Thilikos, & Vigny, 2023]
[Golovach, Stamoulis, & Thilikos, 2023]
[Fomin, Golovach, Sau, Stamoulis, & Thilikos, 2023]
[Pilipczuk, Schirrmacher, Siebertz, Toruńczyk, & Vigny, 2022]
[Schirrmacher, Siebertz, & Vigny, 2022]
[Nešetřil, Ossona de Mendez, & Siebertz, 2022]
[Grange, 2021]
[Berkholz, Keppeler, & Schweikardt, 2018]
[Grohe & Schweikardt, 2018]
[van den Heuvel, Kreutzer, Pilipczuk, Quiroz, Rabinovich, & Siebertz, 2017]

*What about “elementarily-**FPT**”?*

## “Elementarily-FPT” programme

## “Elementarily-FPT” programme

*What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?*

## "Elementarily-FPT" programme

*What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?*

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}}_{\text{height } g(|\varphi|)} \cdot |G|^c, \text{ for some constant } c \geq 1,$$

## “Elementarily-FPT” programme

*What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?*

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}}_{\text{height } g(|\varphi|)} \cdot |G|^c, \text{ for some constant } c \geq 1,$$

even for the class $\mathcal{T}$ of trees. [Frick & Grohe, 2002]

# “Elementarily-FPT” programme

*What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?*

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}}_{\text{height } g(|\varphi|)} \cdot |G|^c, \text{ for some constant } c \geq 1,$$

even for the class $\mathcal{T}$ of trees. [Frick & Grohe, 2002]

**Task:** *Improve the (parametric) dependence on $|\varphi|$ in the running time.*

FO MODEL CHECKING (ON $\mathcal{C}$)
Input: a first-order formula $\varphi$ and a graph $G \in \mathcal{C}$
Question: $G$ satisfies $\varphi$?

## “Elementarily-FPT” programme

*What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?*

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}}_{\text{height } g(|\varphi|)} \cdot |G|^c, \text{ for some constant } c \geq 1,$$

even for the class $\mathcal{T}$ of trees. [Frick & Grohe, 2002]

**Task:** *Improve the (parametric) dependence on $|\varphi|$ in the running time.*

FO MODEL CHECKING (ON $\mathcal{C}$)
Input: a first-order formula $\varphi$ and a graph $G \in \mathcal{C}$
Question: $G$ satisfies $\varphi$?

**Meta-parameter**: $h_{\mathcal{C}}$

## "Elementarily-FPT" programme

*What is the (parametric) dependence on $|\varphi|$ in the running time of a model checking algorithm?*

$$\underbrace{2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}}_{\text{height } g(|\varphi|)} \cdot |G|^c, \text{ for some constant } c \geq 1,$$

even for the class $\mathcal{T}$ of trees. [Frick & Grohe, 2002]

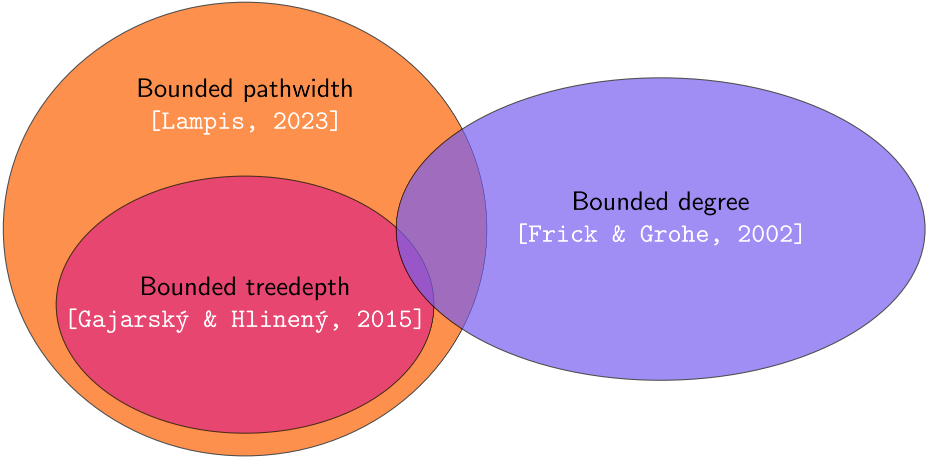
**Task:** *Improve the (parametric) dependence on $|\varphi|$ in the running time.*

FO MODEL CHECKING (ON $\mathcal{C}$)
Input: a first-order formula $\varphi$ and a graph $G \in \mathcal{C}$
Question: $G$ satisfies $\varphi$?

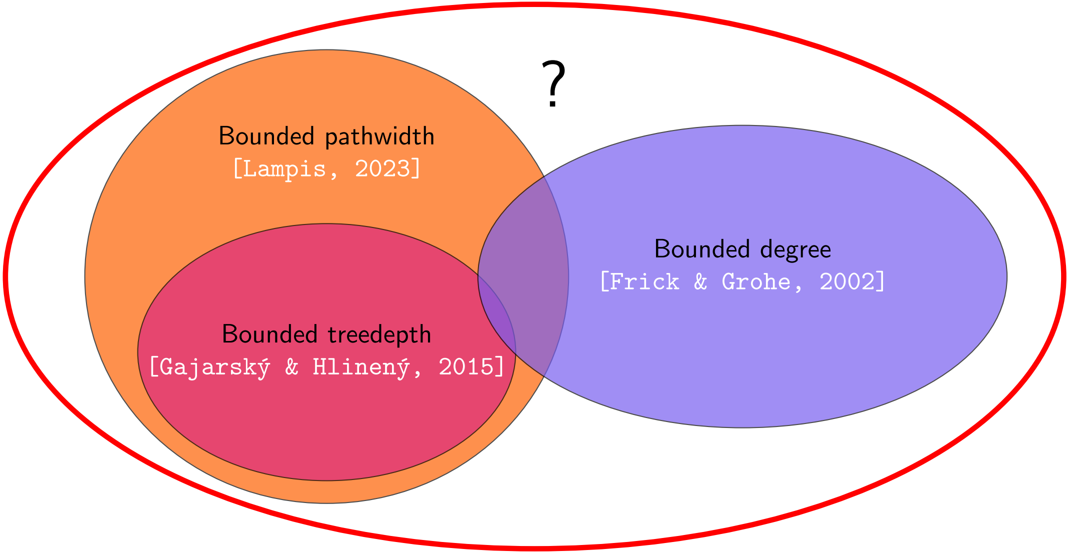
**Meta-parameter**: $h_{\mathcal{C}}$

**Elementarily-FPT**: running time $\underbrace{2^{2^{\cdot^{\cdot^{2^{|\varphi|}}}}}}_{\text{height } g(h_{\mathcal{C}})} \cdot |G|^c$

## The map of the elementarily-FPT universe



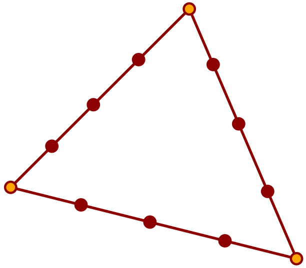
## The map of the elementarily-FPT universe



## Definitions:

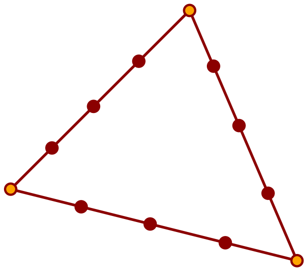
**Definitions:**

- $H^{(\leq r)}$ := replace every edge of $H$ with a path of at most $r$ internal vertices.



**Definitions:**

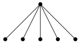
- $H^{(\leq r)}$ := replace every edge of $H$ with a path of at most $r$ internal vertices.
- *$H$ is an $r$-shallow topological minor of $G$*, if $H^{(\leq r)} \subseteq G$.



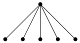
**Definitions:**

- $H^{(\leq r)}$ := replace every edge of $H$ with a path of at most $r$ internal vertices.
- *$H$ is an $r$-shallow topological minor of $G$*, if $H^{(\leq r)} \subseteq G$.
- $\mathrm{TopMinors}_r(\mathcal{C}) := \{H \mid \exists G \in \mathcal{C} : H$ is an $r$-shallow topological minor of $G\}$

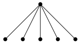
**Definitions:**

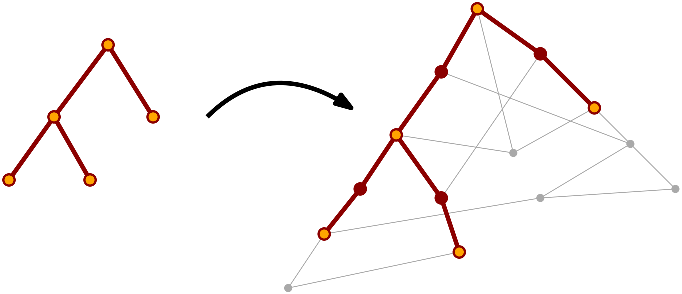
- $H^{(\leq r)}$ := replace every edge of $H$ with a path of at most $r$ internal vertices.
- *$H$ is an $r$-shallow topological minor of $G$*, if $H^{(\leq r)} \subseteq G$.
- $\mathrm{TopMinors}_r(\mathcal{C}) := \{H \mid \exists G \in \mathcal{C} : H$ is an $r$-shallow topological minor of $G\}$
- $\mathcal{T}_d$ := class of all trees of depth $d$. (  has depth 2.)

**Definitions:**

- $H^{(\leq r)}$ := replace every edge of $H$ with a path of at most $r$ internal vertices.
- *$H$ is an $r$-shallow topological minor of $G$*, if $H^{(\leq r)} \subseteq G$.
- $\text{TopMinors}_r(\mathcal{C}) := \{H \mid \exists G \in \mathcal{C} : H$ is an $r$-shallow topological minor of $G\}$
- $\mathcal{T}_d$ := class of all trees of depth $d$. (  has depth 2.)
- The *tree rank* of a graph class $\mathcal{C}$: $\max\{d \in \mathbb{N} \mid \exists r \in \mathbb{N} : \mathcal{T}_d \subseteq \text{TopMinors}_r(\mathcal{C})\}$.

**Definitions:**

- $H^{(\leq r)}$ := replace every edge of $H$ with a path of at most $r$ internal vertices.
- *$H$ is an $r$-shallow topological minor of $G$*, if $H^{(\leq r)} \subseteq G$.
- $\text{TopMinors}_r(\mathcal{C}) := \{H \mid \exists G \in \mathcal{C} : H$ is an $r$-shallow topological minor of $G\}$
- $\mathcal{T}_d$ := class of all trees of depth $d$. (  has depth 2.)
- The *tree rank* of a graph class $\mathcal{C}$: $\max\{d \in \mathbb{N} \mid \exists r \in \mathbb{N} : \mathcal{T}_d \subseteq \text{TopMinors}_r(\mathcal{C})\}$.



## What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.

## What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.
- $\mathcal{T}_d$ has tree rank $d$.

## What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.
- $\mathcal{T}_d$ has tree rank $d$.
- If $\mathcal{C}$ excludes some tree $T$ as a *topological minor*, it has tree rank smaller than the depth of $T$.

## What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.
- $\mathcal{T}_d$ has tree rank $d$.
- If $\mathcal{C}$ excludes some tree $T$ as a *topological minor*, it has tree rank smaller than the depth of $T$.
- $\mathcal{C}$ has bounded degree if and only if $\mathcal{C}$ has tree rank 1.

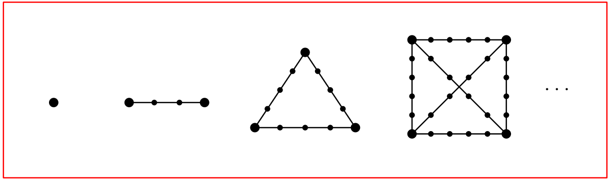
## What is bounded tree rank?

- The class $\mathcal{T}$ of all trees has unbounded tree rank.
- $\mathcal{T}_d$ has tree rank $d$.
- If $\mathcal{C}$ excludes some tree $T$ as a *topological minor*, it has tree rank smaller than the depth of $T$.
- $\mathcal{C}$ has bounded degree if and only if $\mathcal{C}$ has tree rank 1.
- The class $\mathcal{C}$ of graphs of pathwidth $d$ has tree rank exactly $d + 1$.

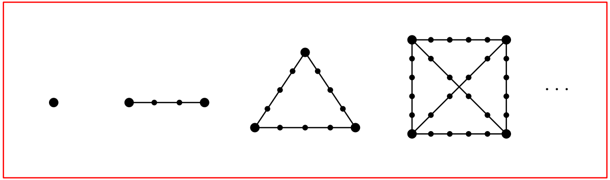
## Is this just excluding a tree as a topological minor?

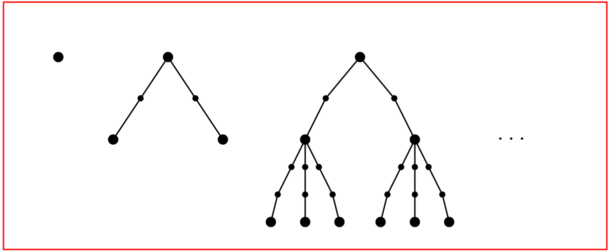
## Is this just excluding a tree as a topological minor? NO

## Is this just excluding a tree as a topological minor? NO

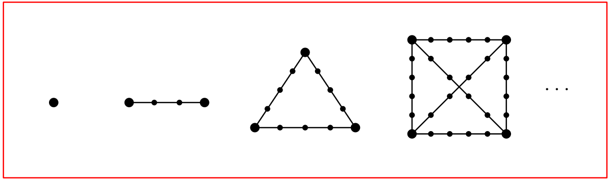


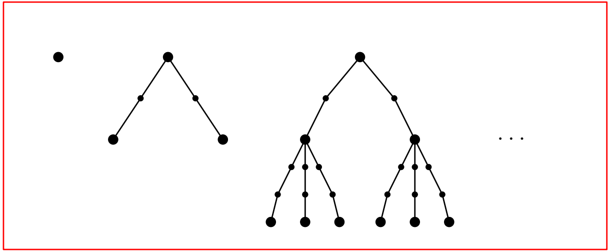
## Is this just excluding a tree as a topological minor? NO



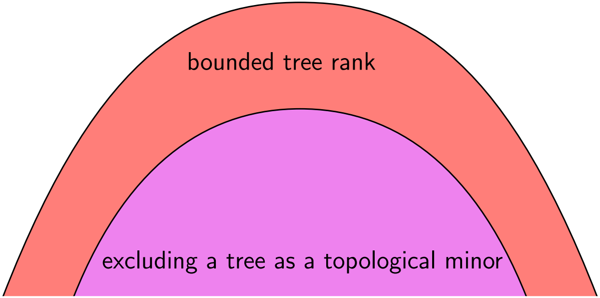


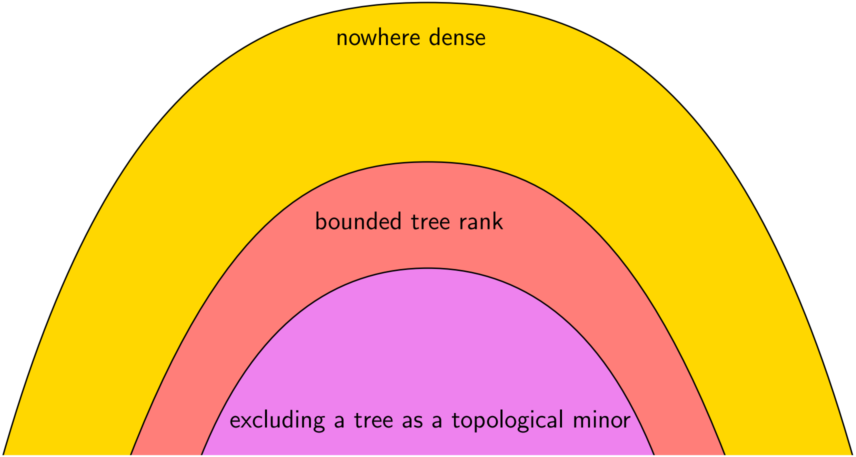
## Is this just excluding a tree as a topological minor? **NO**

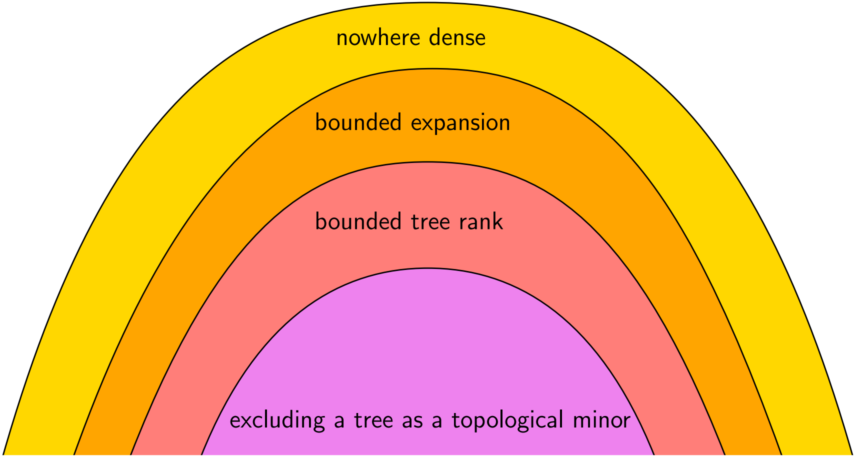


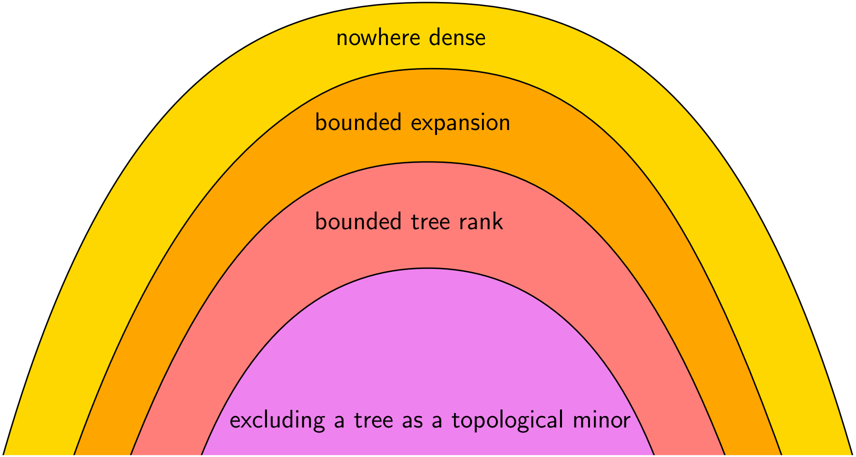


Every tree as a topological minor and tree rank 2


bounded tree rank
excluding a tree as a topological minor


nowhere dense
bounded tree rank
excluding a tree as a topological minor


nowhere dense
bounded expansion
bounded tree rank
excluding a tree as a topological minor




**Fact:** A graph of minimum degree $\delta$ contains every tree on $\delta$ vertices as a subgraph.

bounded tree rank $\Longrightarrow$ bounded degeneracy $\Longrightarrow$ bounded expansion

$T_k^d :=$ tree of depth $d$ and branching $k$.

**Tree rank of $\mathcal{C}$:**
the least number $d \in \mathbb{N}$ such that
for every $r \in \mathbb{N}$ there is $k \in \mathbb{N}$ s.t. **no graph** in $\mathcal{C}$ contains $T_k^{d+1}$ as an $r$-shallow topological minor.

$T_k^d :=$ tree of depth $d$ and branching $k$.

**Tree rank of $\mathcal{C}$:**
the least number $d \in \mathbb{N}$ such that
for every $r \in \mathbb{N}$ there is $k \in \mathbb{N}$ s.t. **no graph** in $\mathcal{C}$ contains $T_k^{d+1}$ as an $r$-shallow topological minor.

**Tree rank of $\mathcal{C}$:**
the least number $d \in \mathbb{N}$ such that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that
for every $r \in \mathbb{N}$, **no graph** in $\mathcal{C}$ contains $T_{f(r)}^{d+1}$ as an $r$-shallow topological minor.

$T_k^d :=$ tree of depth $d$ and branching $k$.

**Tree rank of $\mathcal{C}$:**
the least number $d \in \mathbb{N}$ such that
for every $r \in \mathbb{N}$ there is $k \in \mathbb{N}$ s.t. **no graph** in $\mathcal{C}$ contains $T_k^{d+1}$ as an $r$-shallow topological minor.

**Tree rank of $\mathcal{C}$:**
the least number $d \in \mathbb{N}$ such that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that
for every $r \in \mathbb{N}$, **no graph** in $\mathcal{C}$ contains $T_{f(r)}^{d+1}$ as an $r$-shallow topological minor.

**Elementary tree rank of $\mathcal{C}$:**
the least number $d \in \mathbb{N}$ such that there is an **elementary** function $f : \mathbb{N} \to \mathbb{N}$ such that
for every $r \in \mathbb{N}$, **no graph** in $\mathcal{C}$ contains $T_{f(r)}^{d+1}$ as an $r$-shallow topological minor.

# Elementary FO model checking on sparse classes

## Elementary FO model checking on sparse classes

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

## Elementary FO model checking on sparse classes

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

**Corollary**

If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

---

## Elementary FO model checking on sparse classes

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

**Corollary**

If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

---

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Assume AW[*]$\neq$FPT. Let $\mathcal{C}$ be a monotone graph class.
If FO model checking is elementarily-FPT on $\mathcal{C}$, then $\mathcal{C}$ has bounded tree rank.

## Elementary FO model checking on sparse classes

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

**Corollary**

If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

---

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Assume AW[*]$\neq$FPT. Let $\mathcal{C}$ be a monotone graph class.
If FO model checking is elementarily-FPT on $\mathcal{C}$, then $\mathcal{C}$ has bounded tree rank.

*Almost* complete characterization of elementarily-FPT FO model checking on sparse classes.

# Collapse of FO alternation hierarchy

## Collapse of FO alternation hierarchy

**Lemma**

Let $\mathcal{C}$ be a graph class of tree rank $d$.
Every formula $\varphi$ is equivalent on $\mathcal{C}$ to a formula $\psi$ of alternation rank $3d$.

Also, if $\mathcal{C}$ has *elementary* tree rank $d$, then $|\psi|$ is elementary in $|\varphi|$.

# Collapse of FO alternation hierarchy

**Lemma**

Let $\mathcal{C}$ be a graph class of tree rank $d$.
Every formula $\varphi$ is equivalent on $\mathcal{C}$ to a formula $\psi$ of alternation rank $3d$.

Also, if $\mathcal{C}$ has *elementary* tree rank $d$, then $|\psi|$ is elementary in $|\varphi|$.

$$\exists\forall\exists\exists\forall\exists\forall\exists\ldots\forall\exists \to \underbrace{\boxed{\exists\exists\ldots\exists}\ \boxed{\forall\forall\ldots\forall}\ \cdots\boxed{\exists\exists\ldots\exists}}_{3d \text{ alternations}}$$

# Collapse of FO alternation hierarchy

**Lemma**

Let $\mathcal{C}$ be a graph class of tree rank $d$.
Every formula $\varphi$ is equivalent on $\mathcal{C}$ to a formula $\psi$ of alternation rank $3d$.

Also, if $\mathcal{C}$ has *elementary* tree rank $d$, then $|\psi|$ is elementary in $|\varphi|$.

$$\exists\forall\exists\exists\forall\exists\forall\exists\ldots\forall\exists \to \underbrace{\boxed{\exists\exists\ldots\exists}\ \boxed{\forall\forall\ldots\forall}\ \cdots\boxed{\exists\exists\ldots\exists}}_{3d \text{ alternations}}$$

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Let $\mathcal{C}$ be a <u>monotone</u> graph class. The following are equivalent:

- $\mathcal{C}$ has bounded tree rank
- $\exists k \in \mathbb{N}$ such that for every formula $\varphi$, there is an equivalent (on $\mathcal{C}$) formula $\psi$ of alternation rank $k$.

## Structural characterization of bounded tree rank

*$m$-batched splitter game of radius $r$:*

## Structural characterization of bounded tree rank

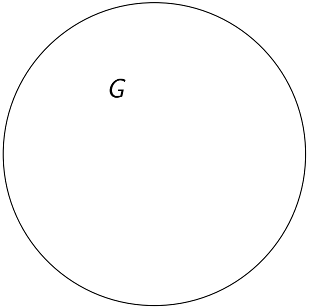
*$m$-batched splitter game of radius $r$:*

Two players: **Splitter** and **Localizer**.

## Structural characterization of bounded tree rank

*m-batched splitter game of radius r:*

Two players: **Splitter** and **Localizer**. In each round of the game:

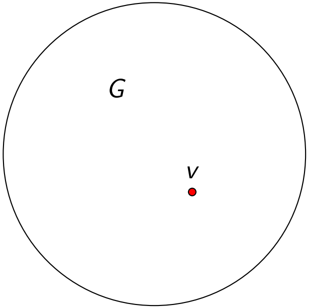


## Structural characterization of bounded tree rank

*m-batched splitter game of radius r:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.

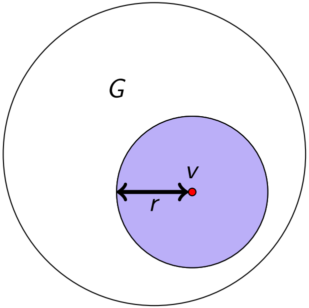


## Structural characterization of bounded tree rank

*m-batched splitter game of radius r:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.

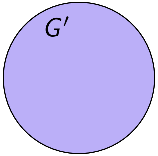


## Structural characterization of bounded tree rank

*m-batched splitter game of radius $r$:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.

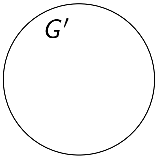


## Structural characterization of bounded tree rank

*m-batched splitter game of radius r:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.
- **Splitter** deletes **at most** $m$ **vertices** from $G'$ and the game continues on the obtained graph.

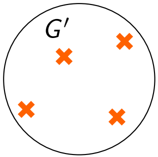


## Structural characterization of bounded tree rank

*$m$-batched splitter game of radius $r$:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.
- **Splitter** deletes **at most $m$ vertices** from $G'$ and the game continues on the obtained graph.

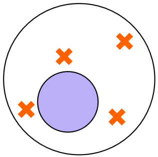


## Structural characterization of bounded tree rank

*$m$-batched splitter game of radius $r$:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.
- **Splitter** deletes **at most** $m$ **vertices** from $G'$ and the game continues on the obtained graph.

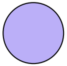


## Structural characterization of bounded tree rank

*m-batched splitter game of radius r:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.
- **Splitter** deletes **at most** $m$ **vertices** from $G'$ and the game continues on the obtained graph.
- If no vertices remain, **Splitter** wins the game.



## Structural characterization of bounded tree rank

*m-batched splitter game of radius r:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.
- **Splitter** deletes **at most** $m$ **vertices** from $G'$
  and the game continues on the obtained graph.
- If no vertices remain, **Splitter** wins the game.

**Lemma**

Let $d \in \mathbb{N}$. The following conditions are equivalent:

(1) $\mathcal{C}$ has (elementary) tree rank $d$,

## Structural characterization of bounded tree rank

*$m$-batched splitter game of radius $r$:*

Two players: **Splitter** and **Localizer**. In each round of the game:

- **Localizer** picks $v \in V(G)$ and restricts to $G' := B^r_G(v)$.
- **Splitter** deletes **at most** $m$ **vertices** from $G'$ and the game continues on the obtained graph.
- If no vertices remain, **Splitter** wins the game.

**Lemma**

Let $d \in \mathbb{N}$. The following conditions are equivalent:

(1) $\mathcal{C}$ has (elementary) tree rank $d$,

(2) There is an (elementary) function $f : \mathbb{N} \to \mathbb{N}$ such that for every $r \in \mathbb{N}$ Splitter wins the $f(r)$-batched splitter game of radius $r$ in at most $d$ rounds, on every $G \in \mathcal{C}$.

## How to do elementary FO model checking?

## How to do elementary FO model checking?

*Compute the "constant alternation rank"-type of the graph,*

using FO model checking algorithm on bounded expansion classes [Dvořák, Král, & Thomas, 2014]

(which is elementarily-FPT for sentences of *constant* alternation rank).

## How to do elementary FO model checking?

*Compute the "constant alternation rank"-type of the graph,*

using FO model checking algorithm on bounded expansion classes [Dvořák, Král, & Thomas, 2014]

(which is elementarily-FPT for sentences of *constant* alternation rank).

The collapse of the FO alternation hierarchy on bounded tree rank classes implies the following:

*If two vertices have the same "constant alternation rank"-type, then they have the same* $\mathsf{q}$*-type.*

## Conclusion

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

**Corollary**

If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

---

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Assume AW[*]$\neq$FPT. Let $\mathcal{C}$ be a monotone graph class.
If FO model checking is elementarily-FPT on $\mathcal{C}$, then $\mathcal{C}$ has bounded tree rank.

*Almost* complete characterization of elementarily-FPT FO model checking on sparse classes.

## Conclusion

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

If $\mathcal{C}$ has bounded elementary tree rank, then FO model checking is elementarily-FPT on $\mathcal{C}$.

**Corollary**

If $\mathcal{C}$ excludes a fixed tree as a topological minor, then FO model checking is elementarily-FPT on $\mathcal{C}$.

---

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Assume AW[*]$\neq$FPT. Let $\mathcal{C}$ be a monotone graph class.
If FO model checking is elementarily-FPT on $\mathcal{C}$, then $\mathcal{C}$ has bounded tree rank.

*Almost* complete characterization of elementarily-FPT FO model checking on sparse classes.

*What about dense classes?*

# Merci!

# Towards dense graph classes

# Towards dense graph classes

**Tree rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that there is an $r \in \mathbb{N}$ such that $\mathcal{T}_d \subseteq \mathsf{TopMinors}_r(\mathcal{C})$.

## Towards dense graph classes

**Tree rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that there is an $r \in \mathbb{N}$ such that $\mathcal{T}_d \subseteq \text{TopMinors}_r(\mathcal{C})$.

**Rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that $\mathcal{C}$ transduces $\mathcal{T}_d$.

## Towards dense graph classes

**Tree rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that there is an $r \in \mathbb{N}$ such that $\mathcal{T}_d \subseteq \text{TopMinors}_r(\mathcal{C})$.

**Rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that $\mathcal{C}$ transduces $\mathcal{T}_d$.

**Conjecture:**
A hereditary graph class $\mathcal{C}$ has elementarily-FPT model checking if and only if it has bounded rank.

## Towards dense graph classes

**Tree rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that there is an $r \in \mathbb{N}$ such that $\mathcal{T}_d \subseteq \mathrm{TopMinors}_r(\mathcal{C})$.

**Rank of $\mathcal{C}$:**
the largest number $d \in \mathbb{N}$ such that $\mathcal{C}$ transduces $\mathcal{T}_d$.

**Conjecture:**
A hereditary graph class $\mathcal{C}$ has elementarily-FPT model checking if and only if it has bounded rank.

**Conjecture:**
Let $\mathcal{C}$ be a hereditary graph class.
$\mathcal{C}$ has bounded rank $\iff \exists k \in \mathbb{N}$ such that every $\varphi$ is equivalent on $\mathcal{C}$ to a $\psi$ of alternation rank $k$.

A graph class $\mathcal{C}$ is weakly sparse if it avoids some biclique as a subgraph.

A graph class $\mathcal{C}$ is weakly sparse if it avoids some biclique as a subgraph.

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Let $\mathcal{C}$ be a weakly sparse graph class. $\mathcal{C}$ has bounded tree rank $\iff$ $\mathcal{C}$ has bounded rank.

A graph class $\mathcal{C}$ is weakly sparse if it avoids some biclique as a subgraph.

**Theorem** [Gajarský, Pilipczuk, Sokołowski, *S.*, Toruńczyk, 2023]

Let $\mathcal{C}$ be a weakly sparse graph class. $\mathcal{C}$ has bounded tree rank $\iff$ $\mathcal{C}$ has bounded rank.

**Conjecture:**
A hereditary graph class $\mathcal{C}$ has elementarily-FPT model checking if and only if it has bounded rank.

**Conjecture:**
Let $\mathcal{C}$ be a hereditary graph class.
$\mathcal{C}$ has bounded rank $\iff$ $\exists k \in \mathbb{N}$ such that every $\varphi$ is equivalent on $\mathcal{C}$ to a $\psi$ of alternation rank $k$.