Exact Algorithms and Lowerbounds for Multiagent Pathfinding

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Nikolaos Melissinos [1] Michal Opler [1]

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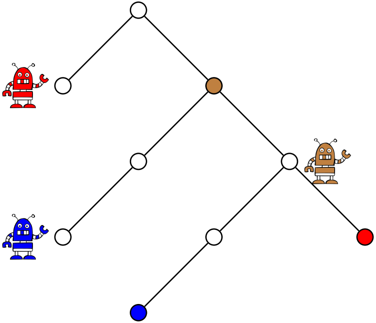
JGA 2023

Multiagent Pathfinding



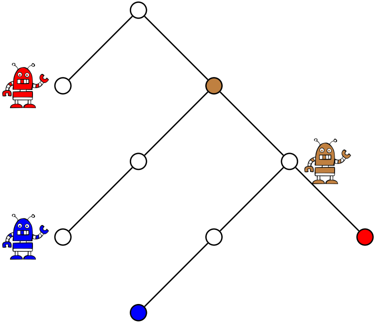
Round 0

The problem:

- Each robot wants to reach its colour
- Move in parallel
- Centralised decisions
- **Two versions:** swap or not

Question:
What is the **makespan** = minimum number of rounds?

Multiagent Pathfinding



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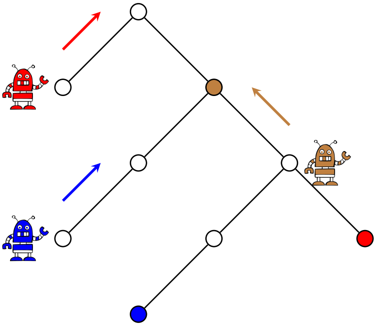
→ Here, at least 5 rounds for the **blue** robot

Multiagent Pathfinding - Swaps allowed



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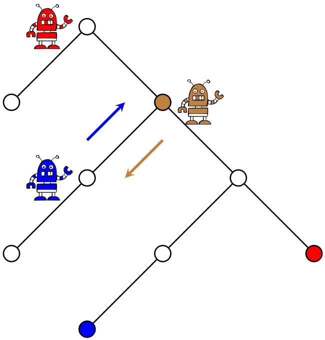

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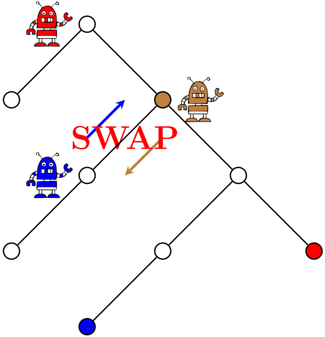

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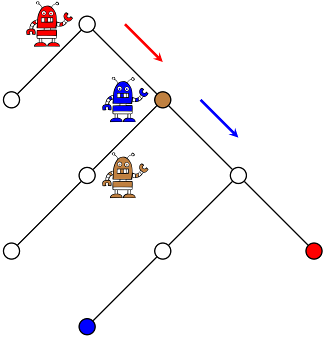

Round 2

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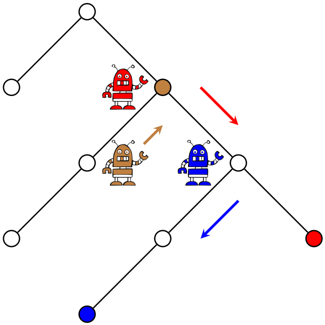

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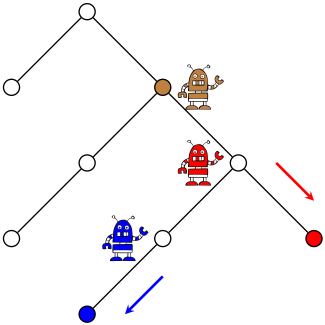

Round 4

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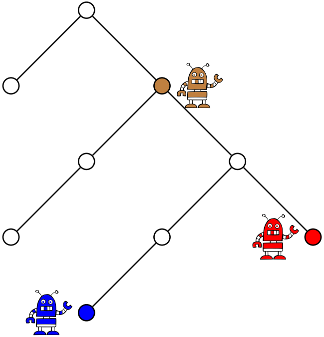

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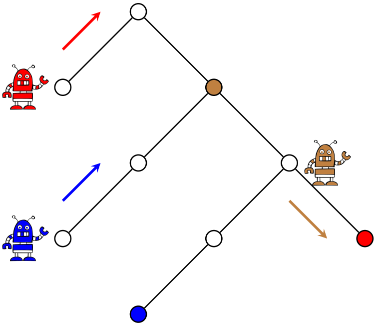

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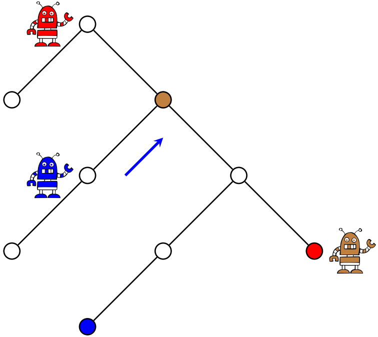

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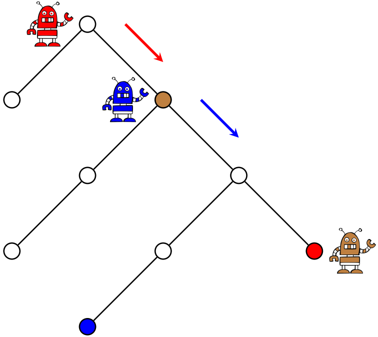

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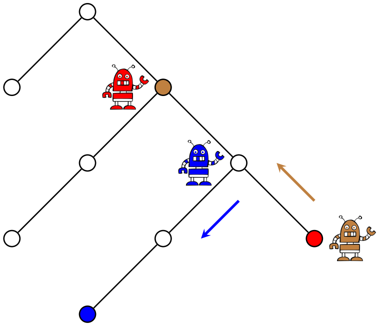

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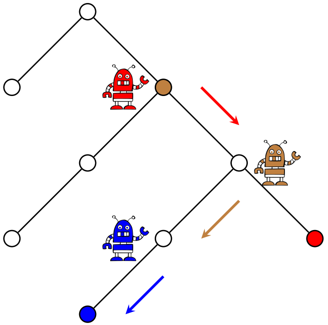

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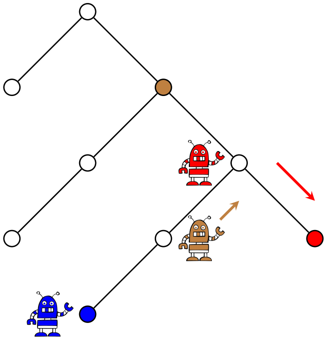

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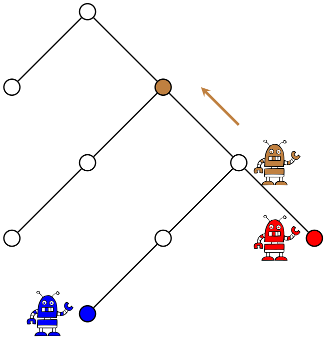

Round 6

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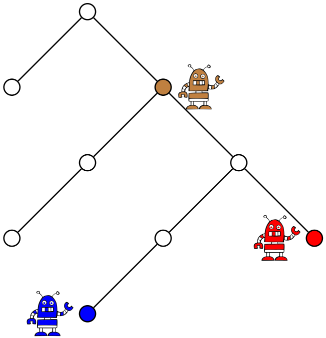

Round 7

Makespan ≤ 7

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Question:
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What is already known?

The problem is hard

- NP-complete (2010, Surynek),
- even on planar graphs (2019, Yu)

Heuristics (2019, Stern)

- A^*-based (1968, Hart, Nilsson, Raphael)
- SAT-based (2017, Surynek et al.)
- Scheduling (2018, Barták, Švancara, Vlk)

Theorem (2023, Eiben, Ganian, Kanj)

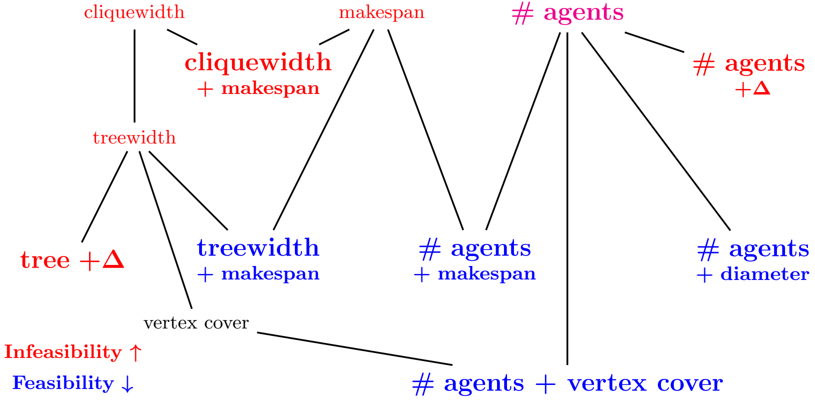
When allowing swaps, deciding if makespan ≤ 26 remains NP-complete even when G is planar and $\Delta(G) = 4$.

Similar problems

- Same problem but **sequential** moves of the robots (1984, Kornhauser, Miller, Spirakis)
- One robot **on each vertex**: Token swapping (2022, Aichholzer et al.), (2018, Bonnet, Miltzow, Rzazewski)

What we did

Studied parameterised complexity of the problem:



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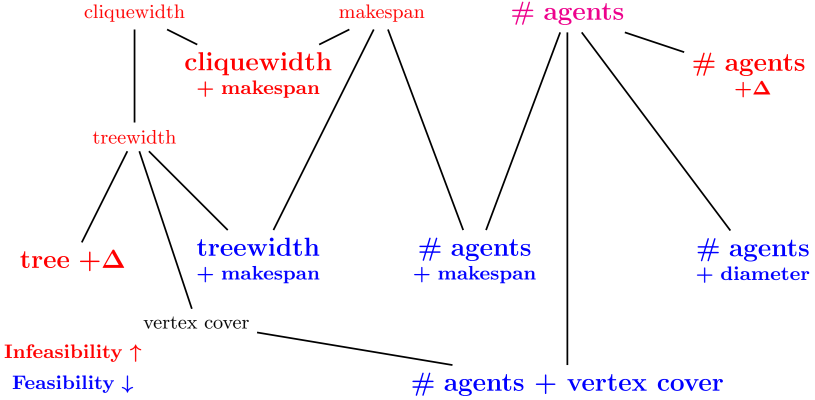
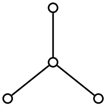


Theorem

When (not resp.) allowing swaps, deciding if makespan ≤ 3 (≤ 2 resp.) remains NP-complete even when G is planar and $\Delta(G) = 4$ ($\Delta(G) = 5$).

Main tool for polynomial algorithms

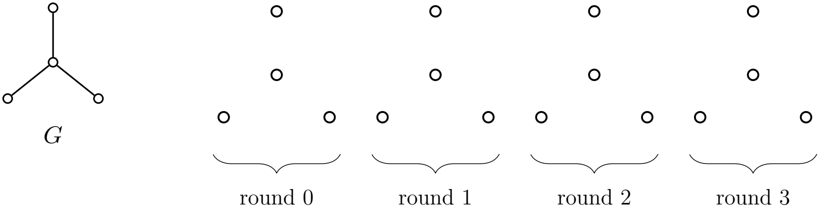
Time-expanded graph



G

Time expanded graph with 3 layers: $G_T(3)$

Time-expanded graph



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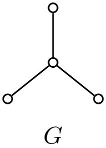

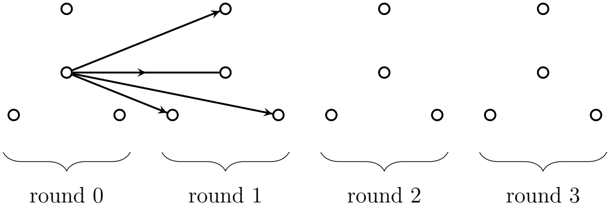

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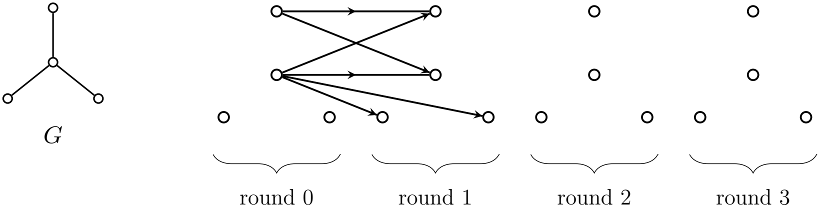

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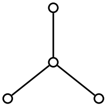

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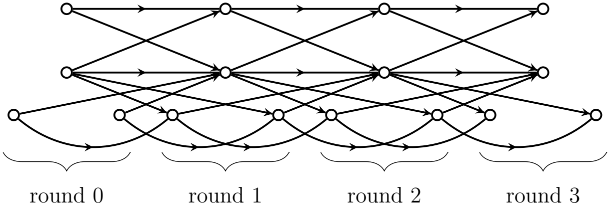

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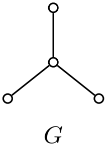
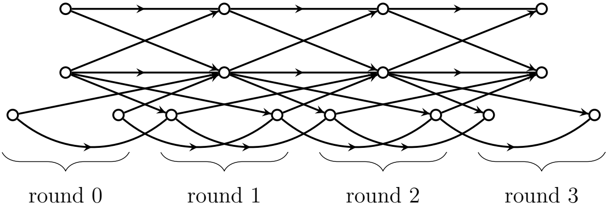


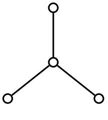


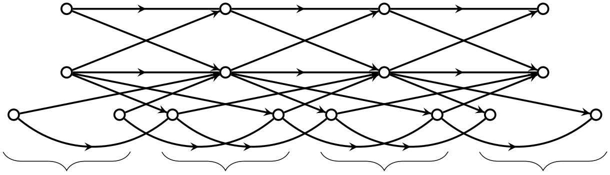
Time expanded graph with 3 layers: $G_T(3)$

Ingredients:

1. Given G, starting and ending positions s_i, t_i, $1 \leq i \leq k$, the makespan is ℓ iff there exist k VERTEX-DISJOINT PATHS from the s_i's to the t_i's in $G_T(\ell)$.

Time-expanded graph





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2. FPT algorithm for k VERTEX-DISJOINT PATHS parameterised by treewidth (1994, Scheffler).
3. FPT algorithm for k VERTEX-DISJOINT PATHS parameterised by $k + \ell$ (2011, Golovach and Thilikos).

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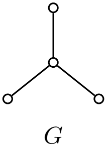

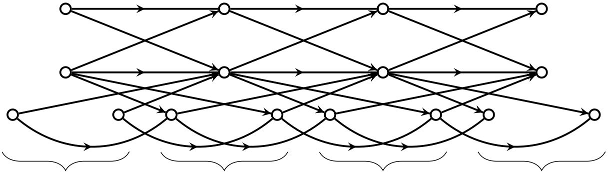

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Careful: $G_T(\ell)$ has treewidth bounded by $tw(G) + \ell$.

NP-hardness for trees

Multiagent Pathfinding on Trees

Theorem

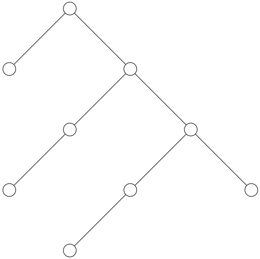
When **not** allowing swaps, it is NP-hard to compute the makespan of T, even when T is a tree with $\Delta(T) = 5$.

Reduction from Token Swapping:

- same problem as ours, one robot on each vertex
- swaps allowed
- NP-hard for trees (2022, Aichholzer et al.)

Multiagent Pathfinding is hard on Trees

Main idea on graphs of treewidth 2:



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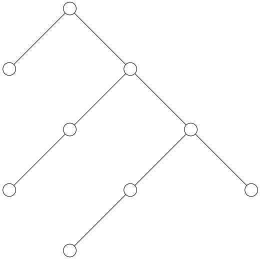

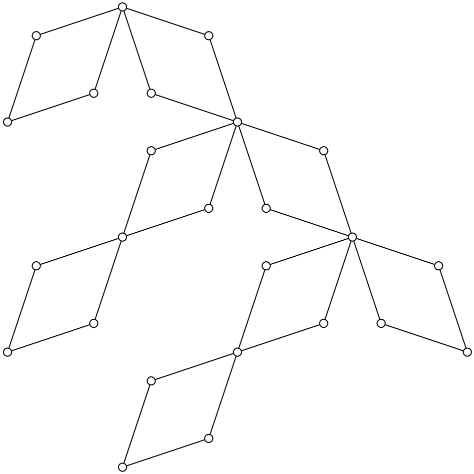

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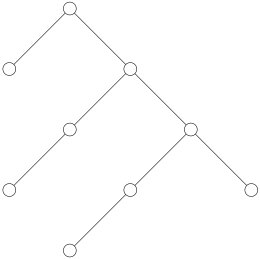

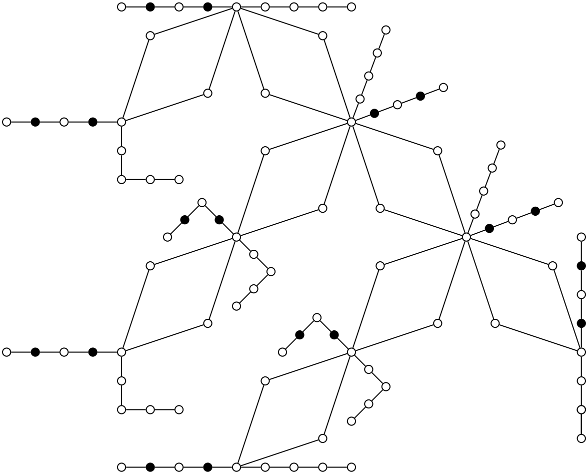

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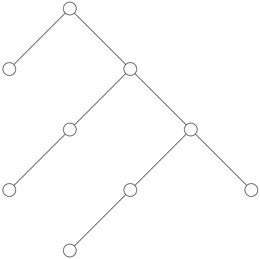

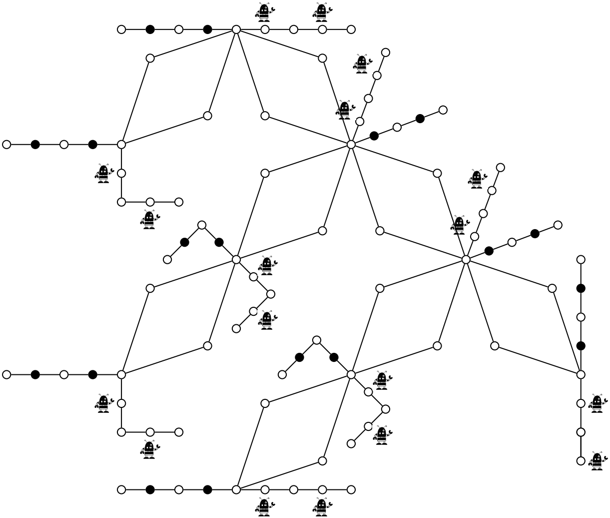

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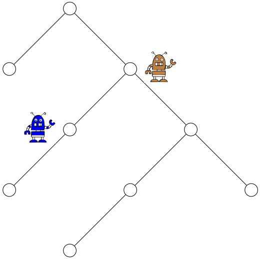

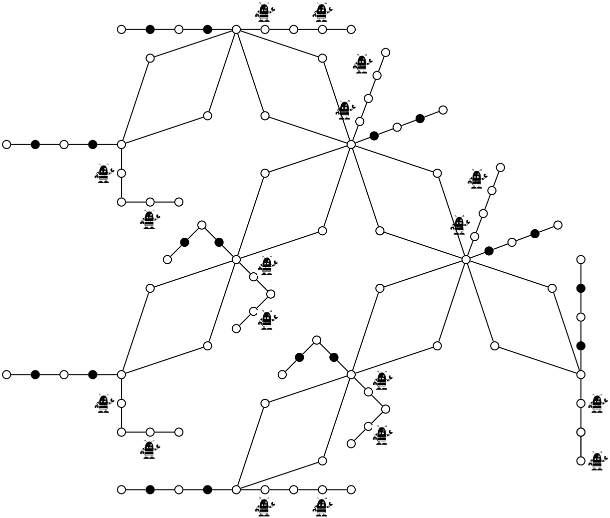

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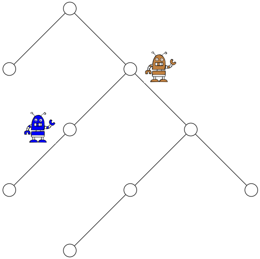

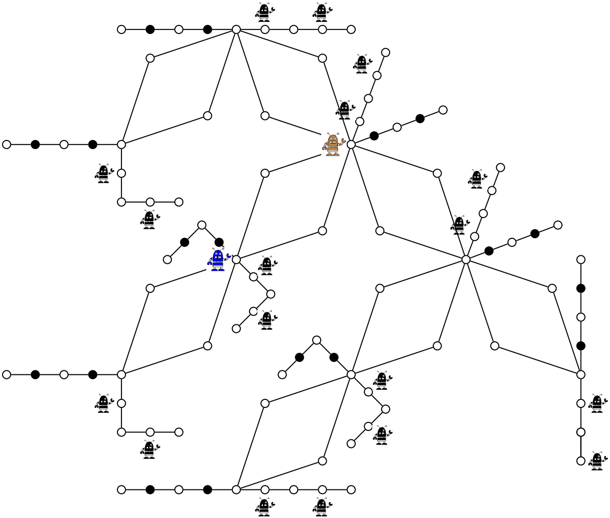

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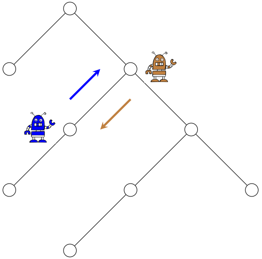

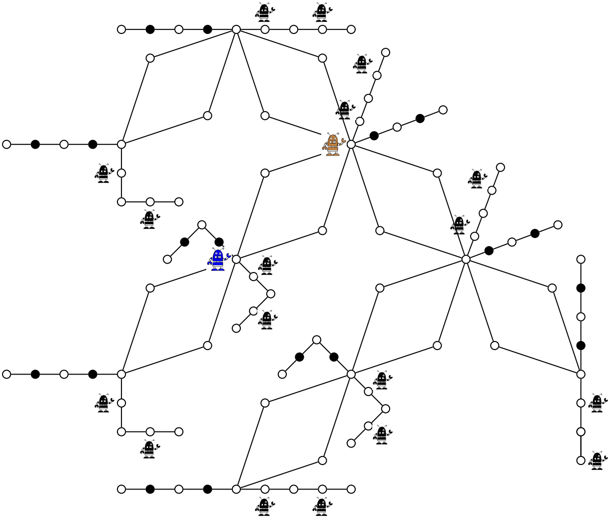

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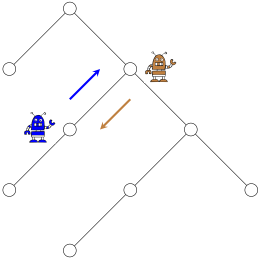

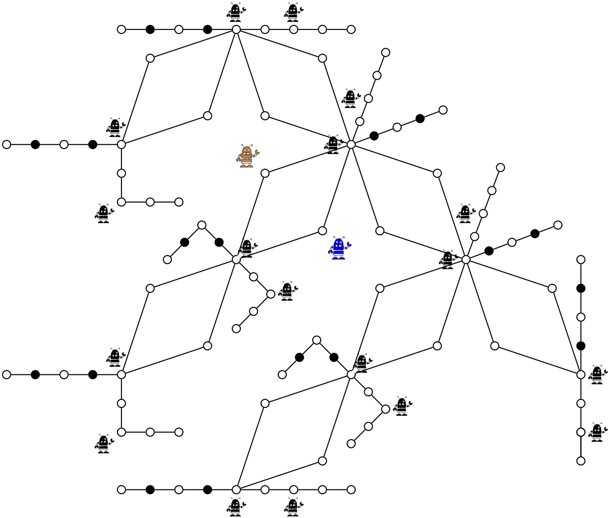

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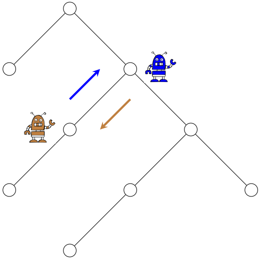

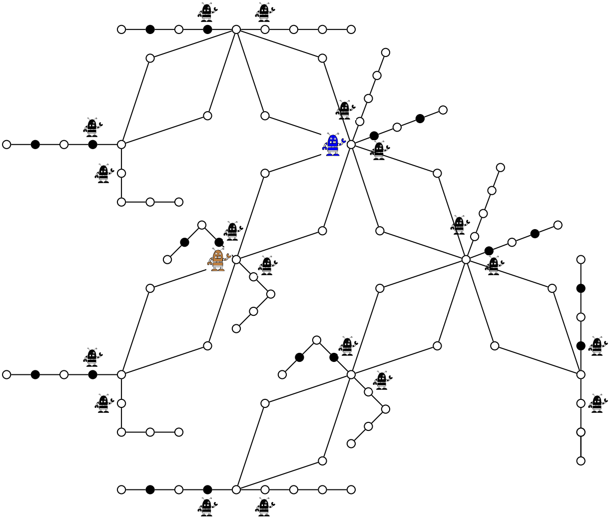

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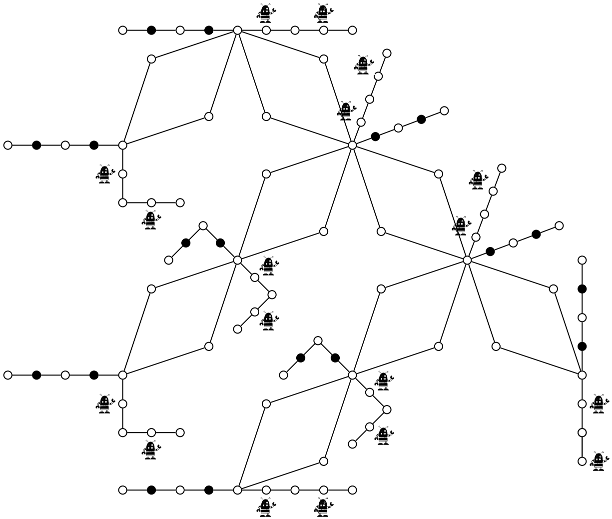

To go to trees:

- Replace rhombuses by complete binary trees of height $\lceil \log(\Delta) + 1 \rceil$
- Carefully adjust the lengths and agents of the extra paths

Conclusion

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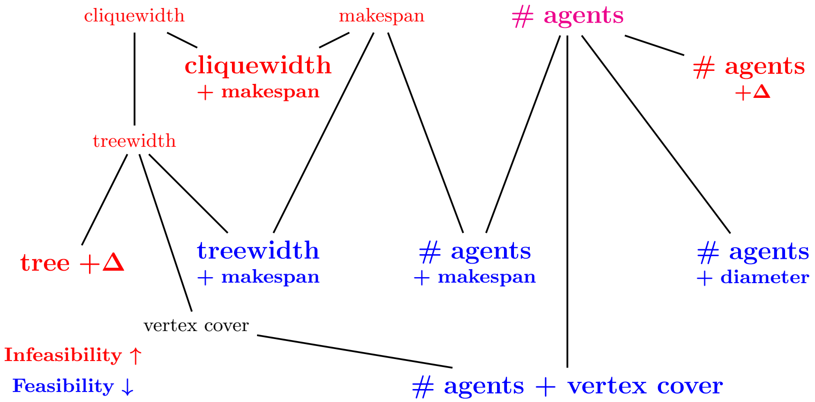
cliquewidth
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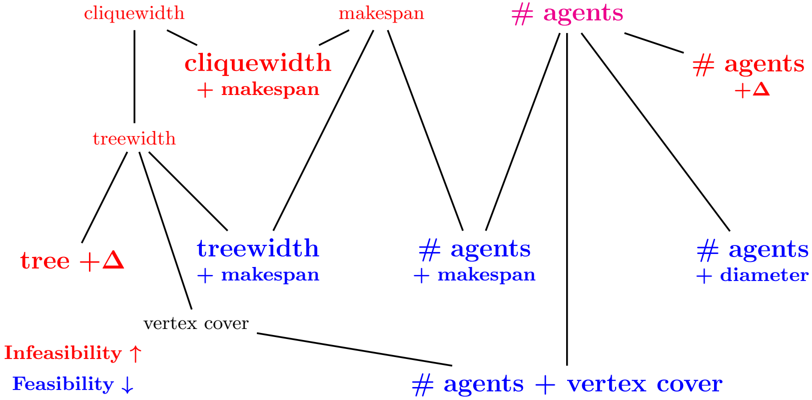
cliquewidth
+ makespan
# agents
+Δ
treewidth
tree +Δ
treewidth
+ makespan
# agents
+ makespan
# agents
+ diameter
vertex cover
Infeasibility ↑
Feasibility ↓
agents + vertex cover

Conclusion



Merci!