Tight (Double) Exponential Bounds for NP-Complete Problems: Treewidth and Vertex Cover Parameterizations

Florent Foucaud, Esther Galby, Liana Khazaliya, Shaohua Li,
Fionn Mc Inerney, Roohani Sharma, and Prafullkumar Tale

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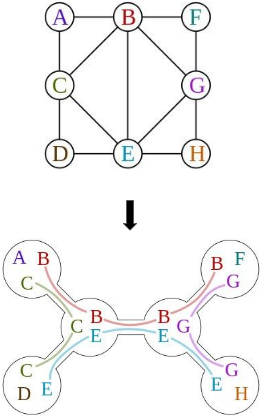
Treewidth

A tree decomposition of a graph $G = (V, E)$ is a tree T with nodes (bags) $X_1, \ldots, X_n$, where each X_i is a subset of V, satisfying

1. $X_1 \cup X_2 \cup \cdots \cup X_n = V$;
2. for all $v \in V$, the bags containing v form a **connected** subtree of T;
3. for all $uv \in E$, there exists a bag containing both u and v.



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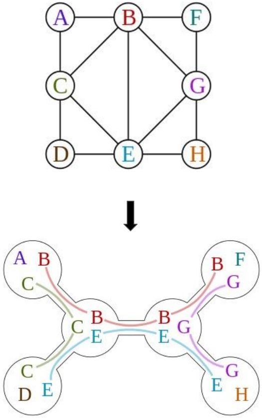
The width of a tree decomposition is the size of the **largest** bag minus one.



Treewidth

The treewidth $\texttt{tw}(G)$ of G is the **minimum** width over all tree decompositions of G.

Treewidth: the **King** of Structural Parameters

Fixed parameter tractable (FPT)

Given a problem Π with input $\mathcal{I}$ and a parameter k, Π is FPT parameterized by k if it can be solved in time $f(k) \cdot |\mathcal{I}|^{O(1)}$, where f is a computable function.

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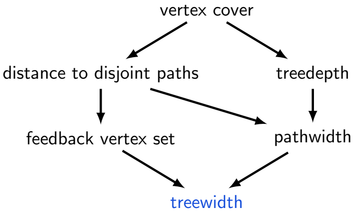
Many NP-hard problems are FPT parameterized by treewidth via dynamic programming on the tree decomposition.

In particular, graph problems expressible in Monadic Second-Order (MSO) logic are FPT parameterized by the treewidth plus the length of the MSO formula [Courcelle, 1990].



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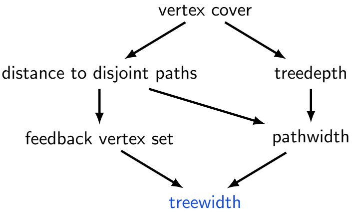

Conditional lower bounds on $f(\mathtt{tw})$ in the FPT algorithms

Exponential Time Hypothesis (ETH) [Impagliazzo, Paturi, 1990]

Roughly, n-variable 3-SAT cannot be solved in time $2^{o(n)}$.

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Conditional lower bounds for $f(\texttt{tw})$ are usually of the form $2^{o(\texttt{tw})}$, or even $2^{o(\texttt{tw} \log \texttt{tw})}$ or $2^{o(\text{poly}(\texttt{tw}))}$.

Rarer results: Unless the ETH fails,

- QSAT (PSPACE-complete) with k alternations admits a lower bound of a tower of exponents of height k in the treewidth of the primal graph [Fichte, Hecher, Pfandler, 2020];
- k-Choosability (Π_2^p-complete) and k-Choosability Deletion (Σ_3^p-complete) admit double- and triple-exponential lower bounds in $\texttt{tw}$, resp. [Marx, Mitsou, 2016];
- $\exists\forall$-CSP (Σ_2^p-complete) admits a double-exponential lower bound in the vertex cover number [Lampis, Mitsou, 2017].

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Common theme: problems are hard for complexity classes higher than NP.

Our results (Part I)

We prove the **first** (conditional) double-exponential lower bounds in the treewidth and vertex cover number for NP-complete problems!

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We develop a **technique** and use it to prove such lower bounds for 3 NP-complete problems:

Theorem [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, Metric Dimension and Geodetic Set do not admit algorithms running in time $2^{f(\mathtt{diam})^{o(\mathtt{tw})}} \cdot n^{O(1)}$, for any computable function f.

Theorem [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

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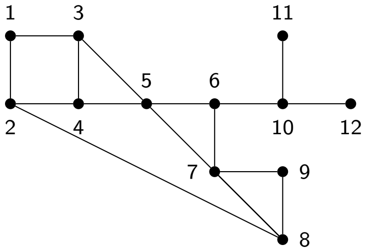
Metric dimension

Metric dimension of a graph $G = (V, E)$ [Slater '75, Harary, Melter '76]

$S \subseteq V$ is a resolving set of G if $\forall u, v \in V$, $\exists z \in S$ with $d(z, u) \neq d(z, v)$.
The **minimum size** of a resolving set of G is the metric dimension of G.



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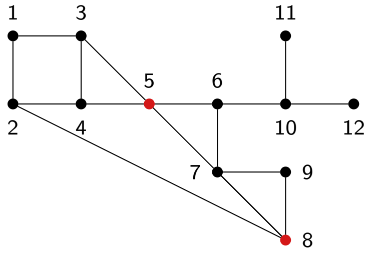

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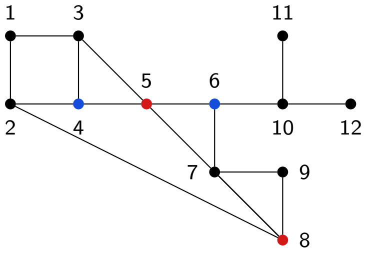

Vertices 4 and 6 are **not** resolved by 5 nor 8.

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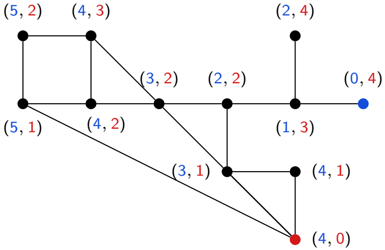

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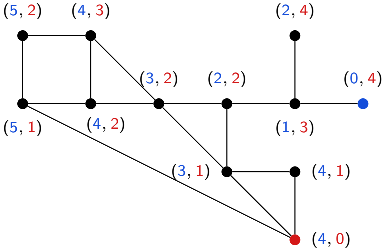

Metric Dimension

Input: an undirected graph $G = (V, E)$ and an integer $k \geq 1$
Question: Is $MD(G) \leq k$?

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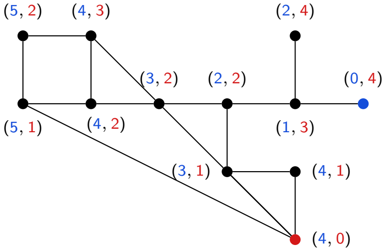


Polynomial-time

Trees	[Slater '75]
Cographs	[Epstein et al '15]
Outerplanar	[Diaz et al '17]

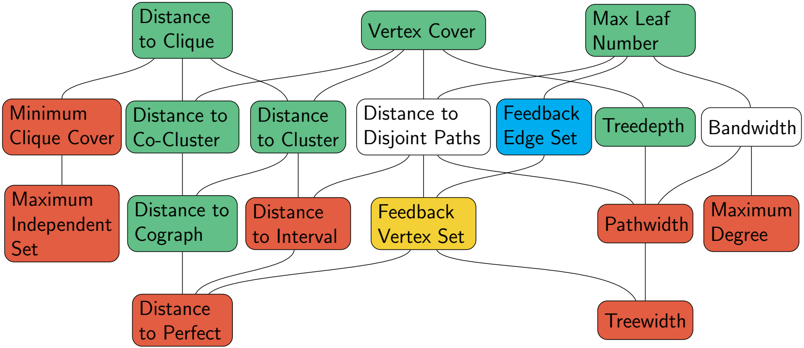
NP-complete

Arbitrary	[Garey, Johnson '79]
Split	[Epstein et al '15]
Bipartite	[Epstein et al '15]
Co-bipartite	[Epstein et al '15]
Planar	[Diaz et al '17]
Interval	[Foucaud et al '17]

- FPT ($f(k) \cdot n^{O(1)}$-time algorithm)
- XP ($n^{f(k)}$-time algorithm)
- W[1]-hard (not FPT unless FPT=W[1])
- para-NP-hard (not XP unless P=NP)

n: size of input

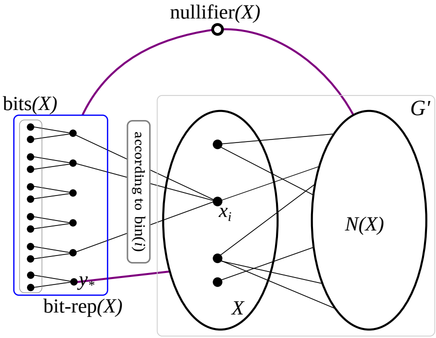
k: size of parameter



NP-hard in graphs of diameter 2 [Foucaud, Mertzios, Naserasr, Parreau, Valicov '17].

NP-hard in graphs of pathwidth 24 [Li, Pilipczuk '22].

First tool: bit-representation gadget



Purple edges represent all possible edges.

Key: bit-rep(X) has size $O(\log |X|)$ and distinguishes each vertex in X from every other vertex in G.

Part I of our technique: reduction from 3-Partitioned-3-SAT

3-Partitioned-3-SAT [Lampis, Melissinos, Vasilakis, 2023]

Input: a 3-CNF formula ϕ with a partition of its variables into 3 disjoint sets X^α, X^β, and X^γ such that $|X^\alpha| = |X^\beta| = |X^\gamma| = n$ and each clause contains at most one variable from each of X^α, X^β, and X^γ.

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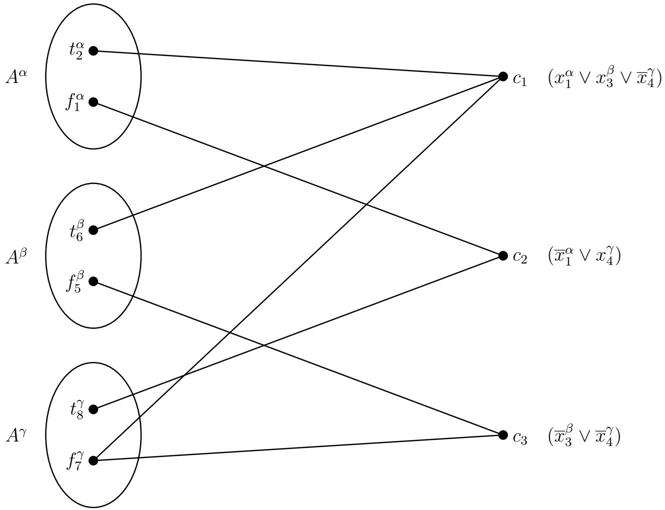
Question: Is ϕ satisfiable?

Theorem [Lampis, Melissinos, Vasilakis, 2023]

Unless the ETH fails, 3-PARTITIONED-3-SAT does not admit an algorithm running in time $2^{o(n)}$.

Part II of our technique: encode SAT with small separator

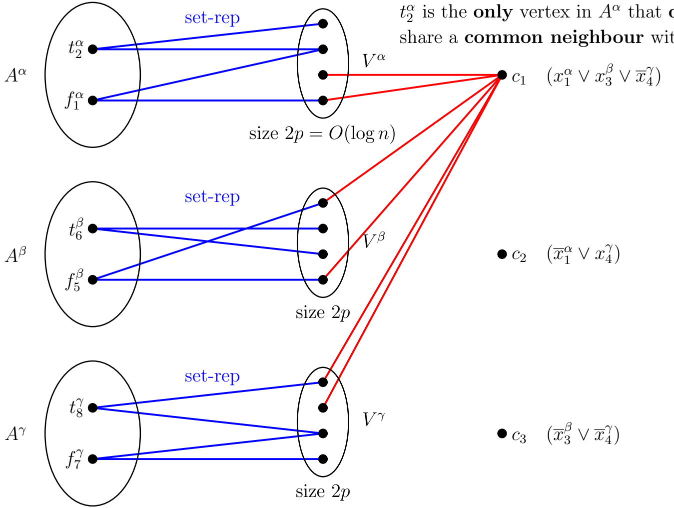
$X^\alpha := \{x_1^\alpha, \ldots, x_n^\alpha\}$, t_{2i}^α represents x_i^α, and f_{2i-1}^α represents $\overline{x}_i^\alpha$.



Part II of our technique: set-representation gadget

Let F_p be the collection of subsets of $\{1, \ldots, 2p\}$ that contain exactly p integers.
No set in F_p is contained in another set in F_p (Sperner family).

There exists $p = O(\log n)$ s.t. $\binom{2p}{p} \geq 2n$. We define a 1-to-1 function set-rep : $\{1, \ldots, 2n\} \rightarrow F_p$.



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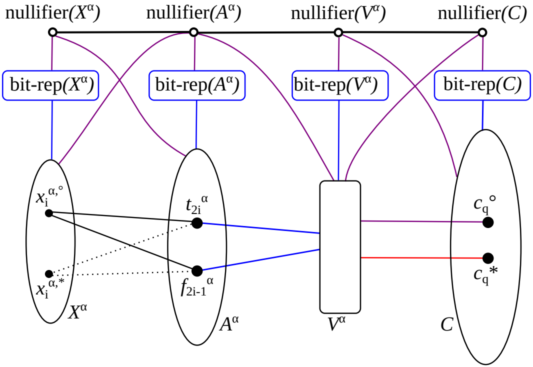

Purple edges represent all possible edges.

Blue edges represent set-rep and red edges are complementary to blue ones.

Budget of $3n$ vertices excluding bit-rep gadgets.

Our results (Part II)

Theorem [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, METRIC DIMENSION and GEODETIC SET do not admit algorithms running in time $2^{o(\mathrm{vc}^2)}$.

Theorem [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, METRIC DIMENSION, GEODETIC SET, and STRONG METRIC DIMENSION do not admit kernelization algorithms outputting kernels with $2^{o(\mathrm{vc})}$ vertices.

We also give matching upper bounds (algorithms and kernels) for our lower bounds.

As far as we know, such kernelization lower bounds were only known for EDGE CLIQUE COVER [Cygan, Pilipczuk, Pilipczuk, 2016] and BICLIQUE COVER [Chandran, Issac, Karrenbauer, 2016].

Further work

Theorem [Chalopin, Chepoi, Mc Inerney, Ratel, 2023]

Unless the ETH fails, Positive Non-Clashing Teaching Dimension for Balls in Graphs does not admit a $2^{2^{o(\mathrm{vc})}} \cdot n^{O(1)}$-time algorithm, nor a kernelization algorithm outputting a kernel with $2^{o(\mathrm{vc})}$ vertices.

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Theorem [Chalopin, Chepoi, Mc Inerney, Ratel, 2023]

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Question: For which classic problems in NP are the best known FPT algorithms parameterized by `tw` or `vc` double-exponential?

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Thanks!