Tight (Double) Exponential Bounds for NP-Complete Problems: Treewidth and Vertex Cover Parameterizations

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Lyon, France, JGA 2023

November 24, 2023
**Treewidth**

A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) with nodes (bags) \( X_1, \ldots, X_n \), where each \( X_i \) is a subset of \( V \), satisfying:

1. \( X_1 \cup X_2 \cup \cdots \cup X_n = V \);
2. For all \( v \in V \), the bags containing \( v \) form a **connected** subtree of \( T \);
3. For all \( uv \in E \), there exists a bag containing both \( u \) and \( v \).

The width of a tree decomposition is the size of the largest bag minus one.

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Many NP-hard problems are FPT parameterized by treewidth via dynamic programming on the tree decomposition.

In particular, graph problems expressible in Monadic Second-Order (MSO) logic are FPT parameterized by the treewidth plus the length of the MSO formula [Courcelle, 1990].
Treewidth: the **King** of Structural Parameters

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Exponential Time Hypothesis (ETH) [Impagliazzo, Paturi, 1990]

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Common theme: problems are hard for complexity classes higher than NP.
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Conditional lower bounds for \( f(tw) \) are usually of the form \( 2^{o(tw)} \), or even \( 2^{o(tw \log tw)} \) or \( 2^{o(poly(tw))} \).

**Rarer results:** Unless the ETH fails,

- QSAT (PSPACE-complete) with \( k \) alternations admits a lower bound of a tower of exponents of height \( k \) in the treewidth of the primal graph [Fichte, Hecher, Pfandler, 2020];

- \( k \)-Choosability (\( \Pi_2^p \)-complete) and \( k \)-Choosability Deletion (\( \Sigma_3^p \)-complete) admit double- and triple-exponential lower bounds in \( tw \), resp. [Marx, Mitsou, 2016];

- \( \exists \forall \)-CSP (\( \Sigma_2^p \)-complete) admits a double-exponential lower bound in the vertex cover number [Lampis, Mitsou, 2017].
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Our results (Part I)

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We develop a **technique** and use it to prove such lower bounds for 3 **NP-complete** problems:

**Theorem** [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, **Metric Dimension** and **Geodetic Set** do not admit algorithms running in time $2^{f(\text{diam})^{o(\text{tw})}} \cdot n^{O(1)}$, for any computable function $f$.

**Theorem** [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, **Strong Metric Dimension** does not admit an algorithm running in time $2^{2^{o(\text{vc})}} \cdot n^{O(1)}$.

We also give **matching upper bounds** (algorithms) for our lower bounds.
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Metric dimension of a graph \( G = (V, E) \) \([\text{Slater '75, Harary, Melter '76}]\)

\( S \subseteq V \) is a \textbf{resolving set} of \( G \) if \( \forall u, v \in V, \exists z \in S \) with \( d(z, u) \neq d(z, v) \).

The \textbf{minimum size} of a resolving set of \( G \) is the \textbf{metric dimension} of \( G \).

Vertices 4 and 6 are not resolved by 5 nor 8.
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![Diagram of a graph with metric dimensions indicated on the vertices]
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**Metric Dimension**

**Input:** an undirected graph $G = (V, E)$ and an integer $k \geq 1$

**Question:** Is $MD(G) \leq k$?
Metric dimension

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Polynomial-time
- Trees [Slater '75]
- Cographs [Epstein et al '15]
- Outerplanar [Diaz et al '17]

NP-complete
- Arbitrary [Garey, Johnson '79]
- Split [Epstein et al '15]
- Bipartite [Epstein et al '15]
- Co-bipartite [Epstein et al '15]
- Planar [Diaz et al '17]
- Interval [Foucaud et al '17]
Parameterized complexity of **Metric Dimension**

- FPT \((f(k) \cdot n^{O(1)})\)-time algorithm
- XP \((n^{f(k)})\)-time algorithm
- W[1]-hard (not FPT unless FPT=W[1])
- para-NP-hard (not XP unless P=NP)

\(n\): size of input  
\(k\): size of parameter

NP-hard in graphs of diameter 2 [Foucaud, Mertzios, Naserasr, Parreau, Valicov '17].

NP-hard in graphs of pathwidth 24 [Li, Pilipczuk '22].
First tool: bit-representation gadget

Purple edges represent all possible edges.

**Key:** bit-rep$(X)$ has size $O(\log |X|)$ and distinguishes each vertex in $X$ from every other vertex in $G$. 
**3-PARTITIONED-3-SAT** [Lampis, Melissinos, Vasilakis, 2023]

**Input**: a 3-CNF formula $\phi$ with a partition of its variables into 3 disjoint sets $X^\alpha$, $X^\beta$, and $X^\gamma$ such that $|X^\alpha| = |X^\beta| = |X^\gamma| = n$ and each clause contains at most one variable from each of $X^\alpha$, $X^\beta$, and $X^\gamma$.

**Question**: Is $\phi$ satisfiable?
Part I of our technique: reduction from 3-PARTITIONED-3-SAT

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**Question:** Is $\phi$ satisfiable?

**Theorem** [Lampis, Melissinos, Vasilakis, 2023]

Unless the ETH fails, 3-PARTITIONED-3-SAT does not admit an algorithm running in time $2^{o(n)}$. 
Part II of our technique: encode SAT with small separator

\[ X^\alpha := \{ x_1^\alpha, \ldots, x_n^\alpha \}, \ t_{2i}^\alpha \text{ represents } x_i^\alpha, \text{ and } f_{2i-1}^\alpha \text{ represents } \overline{x_i}^\alpha. \]
Part II of our technique: set-representation gadget

Let $F_p$ be the collection of subsets of $\{1, \ldots, 2p\}$ that contain exactly $p$ integers. No set in $F_p$ is contained in another set in $F_p$ (Sperner family).

There exists $p = O(\log n)$ s.t. $\binom{2p}{p} \geq 2n$. We define a 1-to-1 function $\text{set-rep} : \{1, \ldots, 2n\} \rightarrow F_p$. 

$t_2^\alpha$ is the only vertex in $A^\alpha$ that does not share a common neighbour with $c_1$. 

$c_1 \quad (x_1^\alpha \lor x_3^\beta \lor \overline{x}_4^\gamma)$

$c_2 \quad (\overline{x}_1^\alpha \lor x_4^\gamma)$

$c_3 \quad (\overline{x}_3^\beta \lor \overline{x}_4^\gamma)$
Lower bound for **Metric Dimension** parameterized by $t_w$

**Theorem** [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, **Metric Dimension** does not admit an algorithm running in time $2^{f(diam)^{o(t_w)}} \cdot n^{O(1)}$, for any computable function $f$.

Purple edges represent all possible edges.  
Blue edges represent set-rep and red edges are complementary to blue ones.  
**Budget** of $3n$ vertices excluding bit-rep gadgets.
Our results (Part II)

Theorem [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, **Metric Dimension** and **Geodetic Set** do not admit algorithms running in time $2^{o(vc^2)}$.

Theorem [Foucaud, Galby, Khazaliya, Li, Mc Inerney, Sharma, Tale, 2023]

Unless the ETH fails, **Metric Dimension**, **Geodetic Set**, and **Strong Metric Dimension** do not admit kernelization algorithms outputting kernels with $2^{o(vc)}$ vertices.

We also give matching upper bounds (algorithms and kernels) for our lower bounds.

As far as we know, such kernelization lower bounds were only known for **Edge Clique Cover** [Cygan, Pilipczuk, Pilipczuk, 2016] and **Biclique Cover** [Chandran, Issac, Karrenbauer, 2016].
Theorem [Chalopin, Chepoi, Mc Inerney, Ratel, 2023]

Unless the ETH fails, **Positive Non-Clashing Teaching Dimension for Balls in Graphs** does not admit a $2^{o(vc)} \cdot n^{O(1)}$-time algorithm, nor a kernelization algorithm outputting a kernel with $2^{o(vc)}$ vertices.
Further work

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**Question:** For which **classic problems** in NP are the best known **FPT algorithms** parameterized by $tw$ or $vc$ **double-exponential**?

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Thanks!