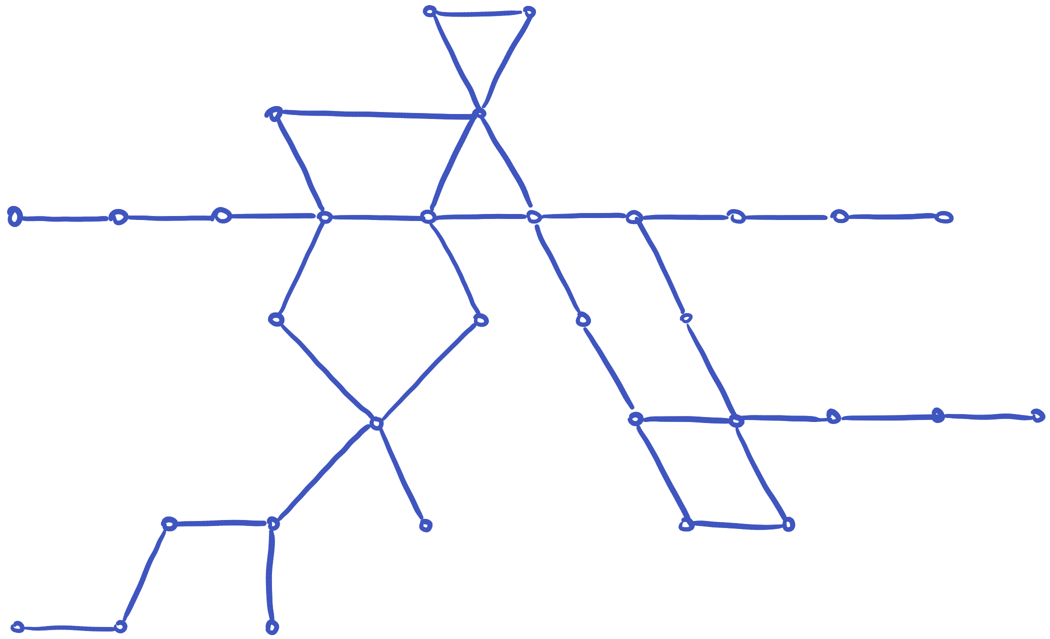
Path Eccentricity

& the consecutive one property

P. Bastide, C. Hilaire, E. Robinson

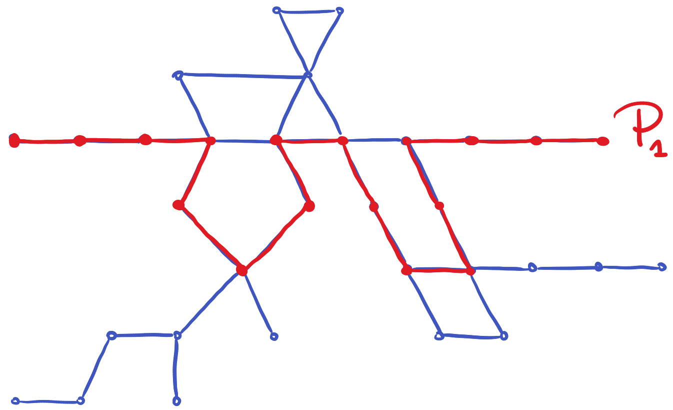
JGA 2023

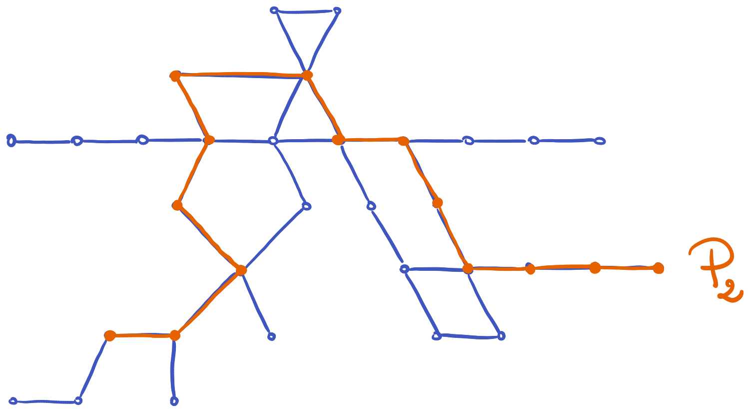
PROBLEM: Find the best localisation for a bus line



→ G is simple, undirected, connected

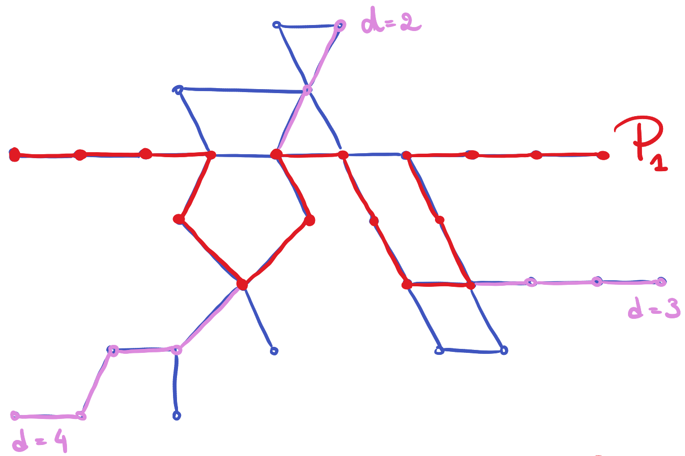
PROBLEM: Find the best localisation for a bus line



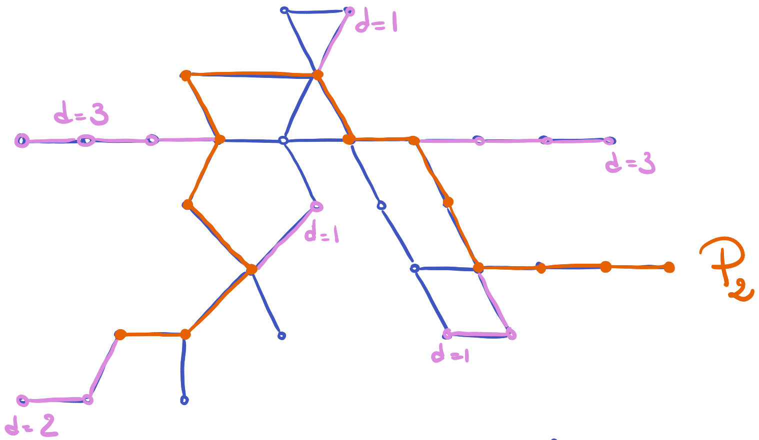


PROBLEM: Find the $\boxed{\text{best}}$ localisation for a bus line

minimizing the maximum distance



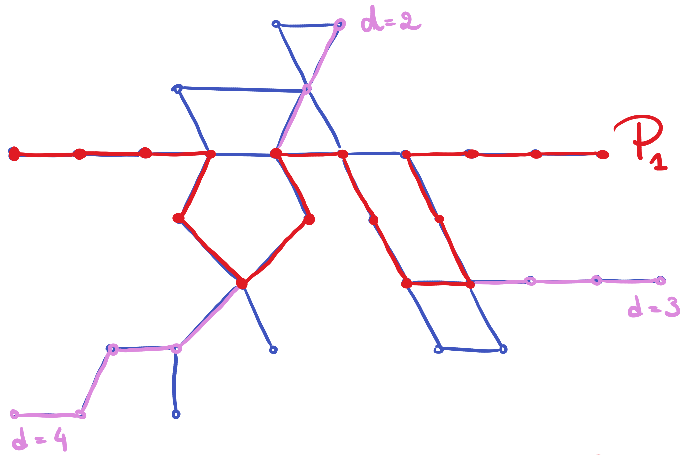
eccentricity of P_1 : 4



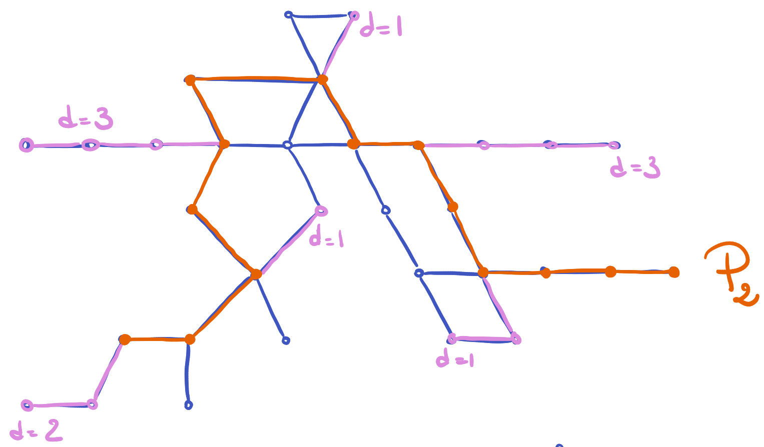
eccentricity of P_2 : 3

PROBLEM : Find the $\boxed{\text{best}}$ localisation for a bus line

minimizing the maximum distance



eccentricity of P_1 : 4



eccentricity of P_2 : 3

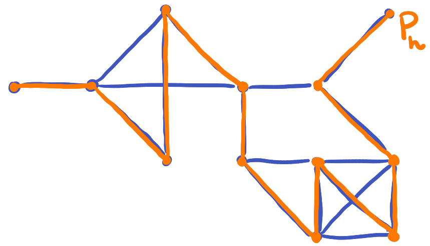
« CENTRAL PATH » := a path with minimal eccentricity

« PATH ECCENTRICITY OF G » := eccentricity of a central path

noted pe(G)

INTERESTING NOTES :

* H a graph. Then $pe(H) = 0$ iff H admits an hamiltonian path.

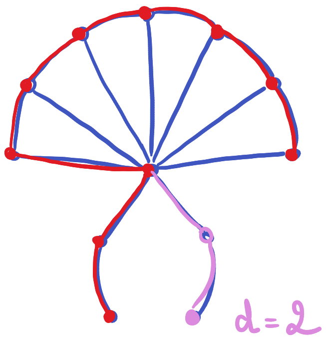


eccentricity P_h : 0

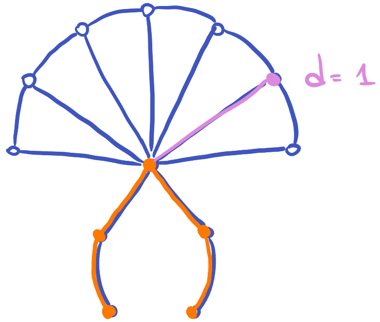
INTERESTING NOTES :

* H a graph. Then $pe(H) = 0$ iff H admits an hamiltonian path.

* G a graph. A longest path in G is not always a central path.



eccentricity of P_1 : 2



eccentricity of P_2 : 1

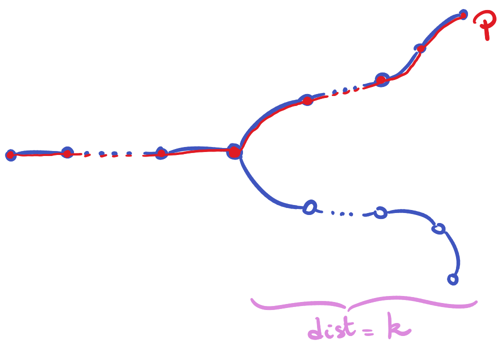
Interesting notes :

* H a graph. Then $pe(H) = 0$ iff H admits an hamiltonian path.

* G a graph. A longest path in G is not always a central path.

* Path eccentricity is unbounded.

k-subdivided claw



eccentricity of P : k

Question of Gómez & Gutiérrez*:

How does the consecutive ones property (or variations of it) influence the path eccentricity of a graph?

* Path eccentricity of graphs. DAM 2023

The consecutive ones property:

A 0-1 matrix is said to have the « consecutive ones property » (noted C1P) if there exists a permutation of its rows that places the 1s consecutively in every column.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{ \underline{has} the C1P}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \text{ \underline{has not} the C1P}$$

THE CONSECUTIVE ONES PROPERTY :

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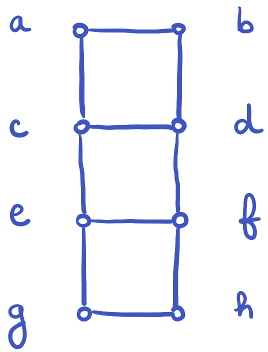
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{has the C1P}$$

$$\xrightarrow[L_2 \leftrightarrow L_4]{} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{has not the C1P}$$

One graph, multiple matrices:

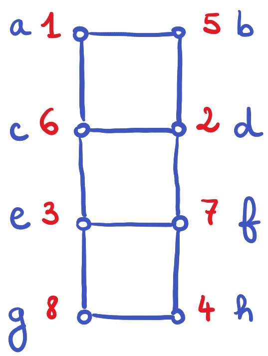
1 Adjacency matrix of G



$$
\begin{array}{c c}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\end{array}
$$

ONE GRAPH, MULTIPLE MATRICES:

1 Adjacency matrix of G has the C1P



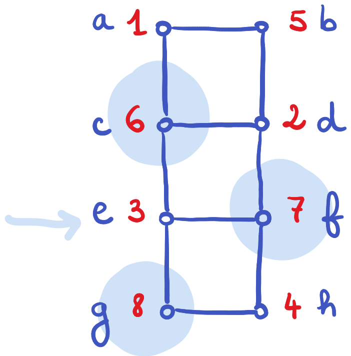
$$
\begin{array}{cc}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{cc} 1 & a \\ 5 & b \\ 6 & c \\ 2 & d \\ 3 & e \\ 7 & f \\ 8 & g \\ 4 & h \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\end{array}
\xrightarrow{\mu}
\begin{array}{cc}
 & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\
\begin{array}{cc} 1 & a \\ 2 & d \\ 3 & e \\ 4 & h \\ 5 & b \\ 6 & c \\ 7 & f \\ 8 & g \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\end{array}
\longrightarrow
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a & d & e & h & b & c & f & g \end{array} \\
\begin{array}{c} a \\ d \\ e \\ h \\ b \\ c \\ f \\ g \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

permutation of the rows:

μ: $a \to 1$ $\quad$ $e \to 3$
$b \to 5$ $\quad$ $f \to 7$
$c \to 6$ $\quad$ $g \to 8$
$d \to 2$ $\quad$ $h \to 4$

One graph, multiple matrices:

① Adjacency matrix of G has the C1P

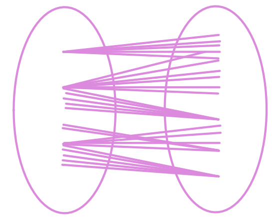


$$
\begin{array}{cc|cccccccc}
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 & & a & d & e & h & b & c & f & g \\
\hline
1 & a & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
2 & d & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
3 & e & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
4 & h & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
5 & b & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & c & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
7 & f & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
8 & g & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

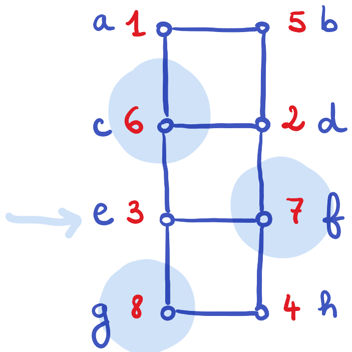
There exists an order of the vertices μ such that the open neighborhood of every vertex is consecutive

$\boxed{\text{G has the « open consecutive ones property » (o-C1P)}}$

ONE GRAPH, MULTIPLE MATRICES :


Biconvex graph

① Adjacency matrix of G has the C1P



$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ a & d & e & h & b & c & f & g \end{array} \\
\begin{array}{c} 1\ a \\ 2\ d \\ 3\ e \\ 4\ h \\ 5\ b \\ 6\ c \\ 7\ f \\ 8\ g \end{array} &
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

There exists an order of the vertices μ such that the open neighborhood of every vertex is consecutive

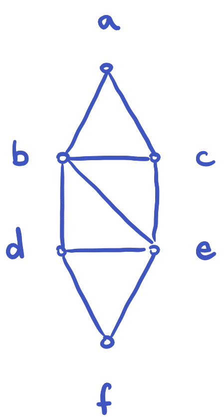
G has the « open consecutive ones property » (o-C1P)

$\overset{B,H,R}{\Longleftrightarrow}$ G is a biconvex graph $\overset{G,G}{\Longrightarrow}$ $pe(G) \leq 1$

One graph, multiple matrices:

① Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

② Augmented adjacency matrix of G has the C1P :

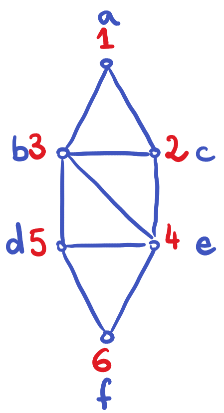


$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 1 & 1 & 0 & 0 & 0 \\ b & 1 & 1 & 1 & 1 & 1 & 0 \\ c & 1 & 1 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 & 1 & 1 \\ e & 0 & 1 & 1 & 1 & 1 & 1 \\ f & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

One graph, multiple matrices:

① Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

② Augmented adjacency matrix of G has the C1P :



$$
\begin{array}{c|c}
 & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}
\xrightarrow{\mu}
\begin{array}{c|c}
 & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\
\begin{array}{c} 1\ a \\ 2\ c \\ 3\ b \\ 4\ e \\ 5\ d \\ 6\ f \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}
\longrightarrow
\begin{array}{c|c}
 & \begin{array}{cccccc} \overset{1}{a} & \overset{2}{c} & \overset{3}{b} & \overset{4}{e} & \overset{5}{d} & \overset{6}{f} \end{array} \\
\begin{array}{c} a \\ c \\ b \\ e \\ d \\ f \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\end{array}
$$

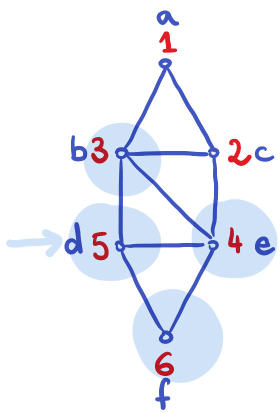
permutation of the rows:

μ: $a \to 1$ $\quad d \to 5$
$b \to 3$ $\quad e \to 4$
$c \to 2$ $\quad f \to 6$

One graph, multiple matrices:

(1) Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

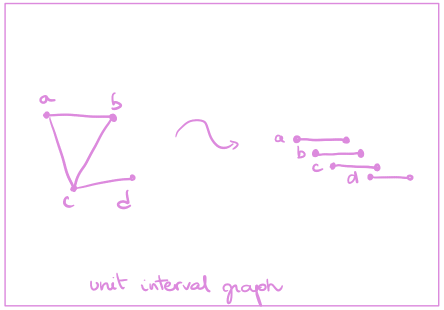
(2) Augmented adjacency matrix of G has the C1P :



$$\begin{array}{cc|cccccc|}
 & & 1 & 2 & 3 & 4 & 5 & 6 \\
 & & a & c & b & e & d & f \\
\hline
1 & a & 1 & 1 & 1 & 0 & 0 & 0 \\
2 & c & 1 & 1 & 1 & 1 & 0 & 0 \\
3 & b & 1 & 1 & 1 & 1 & 1 & 0 \\
4 & e & 0 & 1 & 1 & 1 & 1 & 1 \\
5 & d & 0 & 0 & 1 & 1 & 1 & 1 \\
6 & f & 0 & 0 & 0 & 1 & 1 & 1
\end{array}$$

There exists an order of the vertices such that the closed neighborhood of every vertex is consecutive

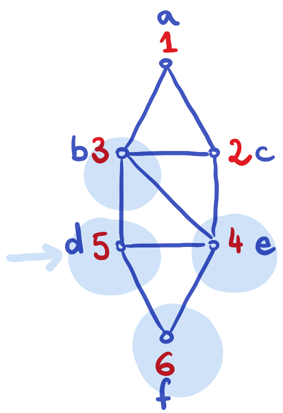
$\boxed{\text{G has the « closed consecutive ones property » (c-C1P)}}$



One graph, multiple matrices:

① Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

② Augmented adjacency matrix of G has the C1P :



$$\begin{array}{c|cccccc} & a & b & c & d & e & f \\ \hline a & 1 & 1 & 1 & 0 & 0 & 0 \\ b & 1 & 1 & 1 & 1 & 1 & 0 \\ c & 1 & 1 & 1 & 0 & 1 & 0 \\ d & 0 & 1 & 0 & 1 & 1 & 1 \\ e & 0 & 1 & 1 & 1 & 1 & 1 \\ f & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

There exists an order of the vertices μ such that the closed neighborhood of every vertex is consecutive

G has the « closed consecutive ones property » (c-C1P)

$\overset{B,H,R}{\Longleftrightarrow}$ G is a unit interval graph $\underset{\text{Stewart '97}}{\overset{\text{Corneil, Olariu,}}{\Longrightarrow}}$ $pe(G) \leq 1$

ONE GRAPH, MULTIPLE MATRICES:

1. Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

2. Augmented adjacency matrix of G has the C1P : G has the closed-C1P, is a unit interval, and $pe(G) \leq 1$

3. Mixed adjacency matrix of G has the C1P :

$$\underbrace{\begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{bmatrix}}_{\substack{\text{adjacency matrix} \\ \text{+ choice of diagonal}}}$$

There exists a choice of 0-1 for the diagonal of the adjacency matrix of G s.t. it has the C1P.

One graph, multiple matrices:

(1) Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

(2) Augmented adjacency matrix of G has the C1P : G has the closed-C1P, is a unit interval, and $pe(G) \leq 1$

(3) Mixed adjacency matrix of G has the C1P :

There exists an order μ on the vertices s.t. for every vertex, either the closed or the open neighborhood is consecutive

There exists a choice of 0-1 for the diagonal of the adjacency matrix of G s.t. it has the C1P.

G has the « mixed consecutive ones property » (m-C1P)

One graph, multiple matrices :

① Adjacency matrix of G has the C1P : G has the open-C1P, is biconvex, and $pe(G) \leq 1$

② Augmented adjacency matrix of G has the C1P : G has the closed-C1P, is a unit interval, and $pe(G) \leq 1$

③ Mixed adjacency matrix of G has the C1P :

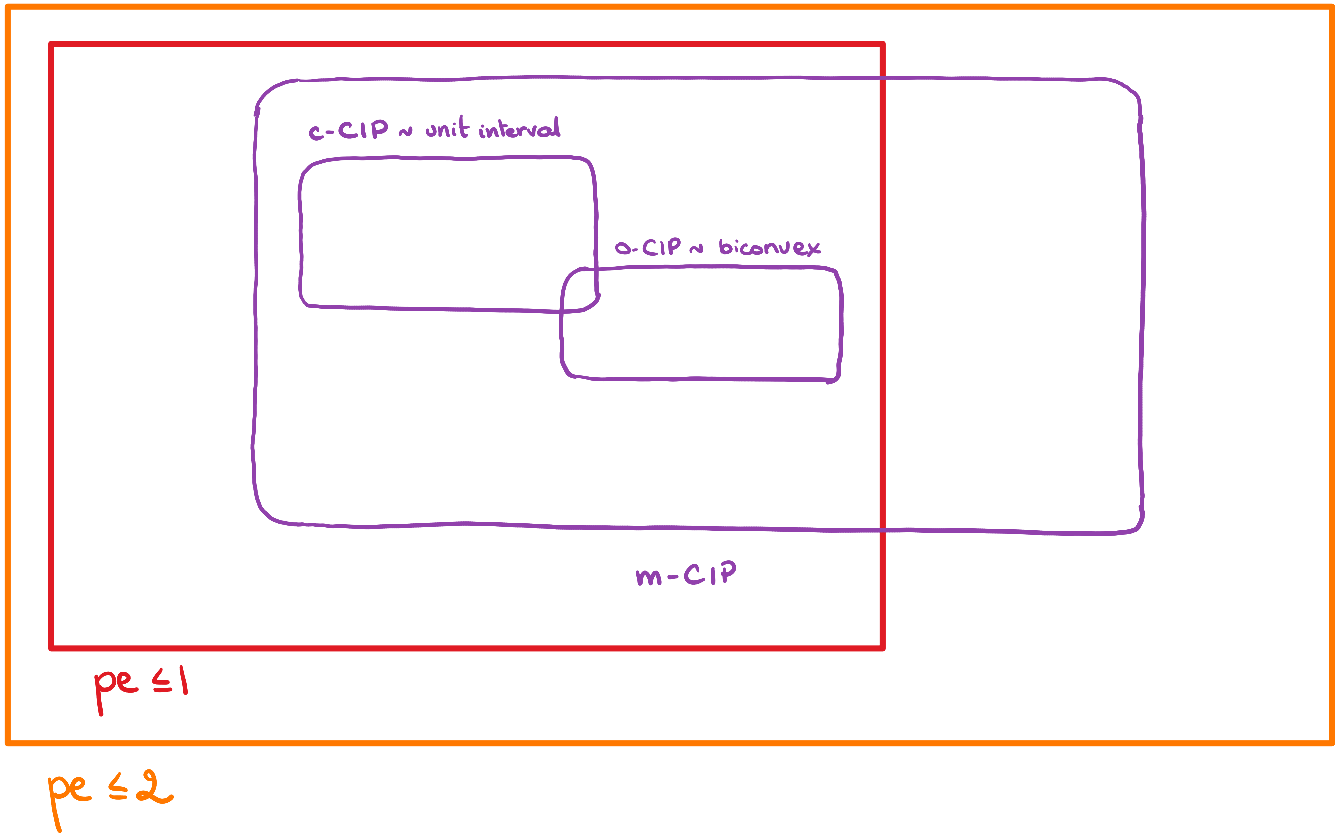
There exists an order μ on the vertices s.t. for every vertex, either the closed or the open neighborhood is consecutive

There exists a choice of 0-1 for the diagonal of the adjacency matrix of G s.t. it has the C1P.

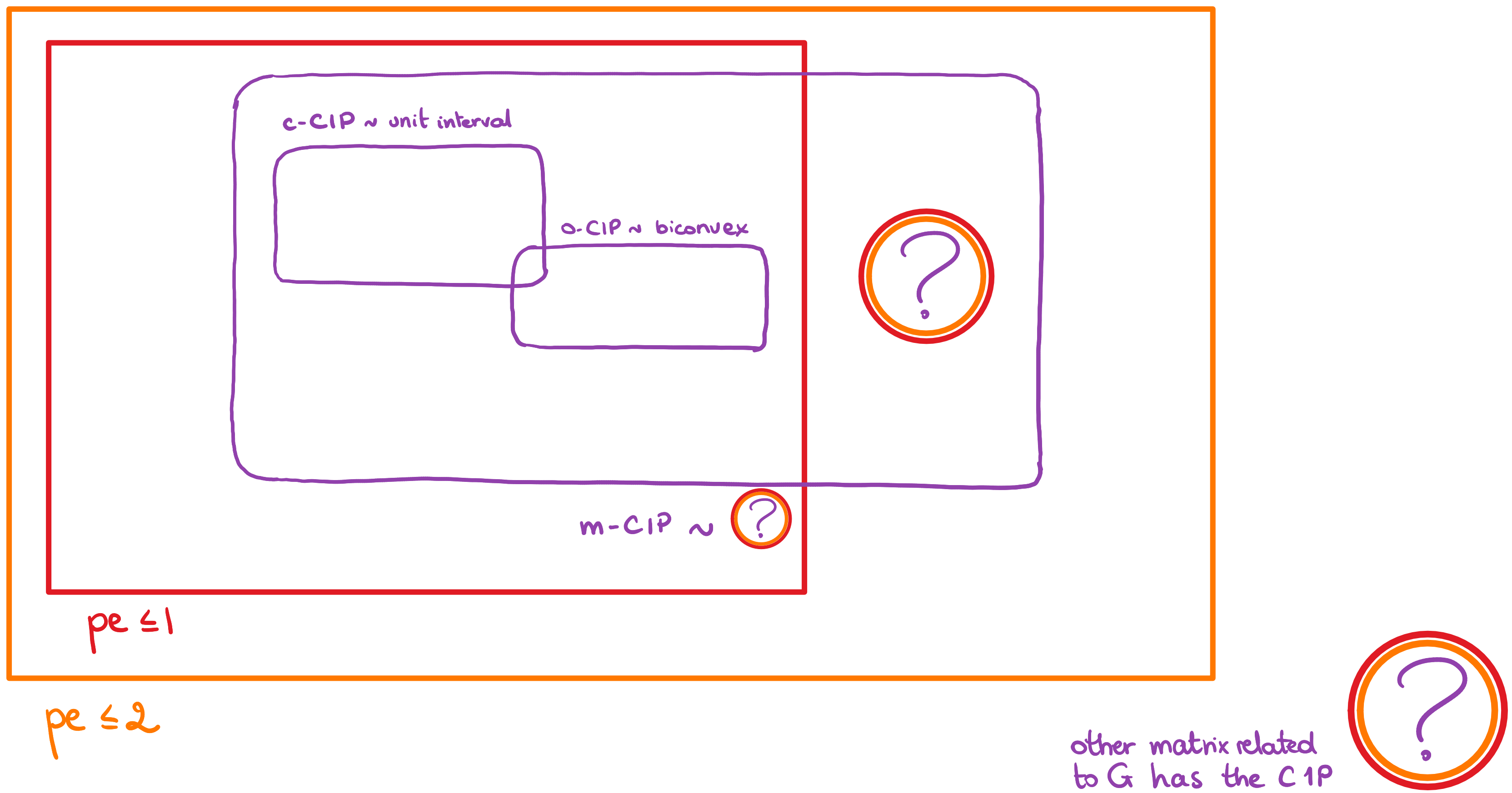
G has the « mixed consecutive ones property » (m-C1P)

$\xRightarrow{B,H,R}$ $pe(G) \leq 2$

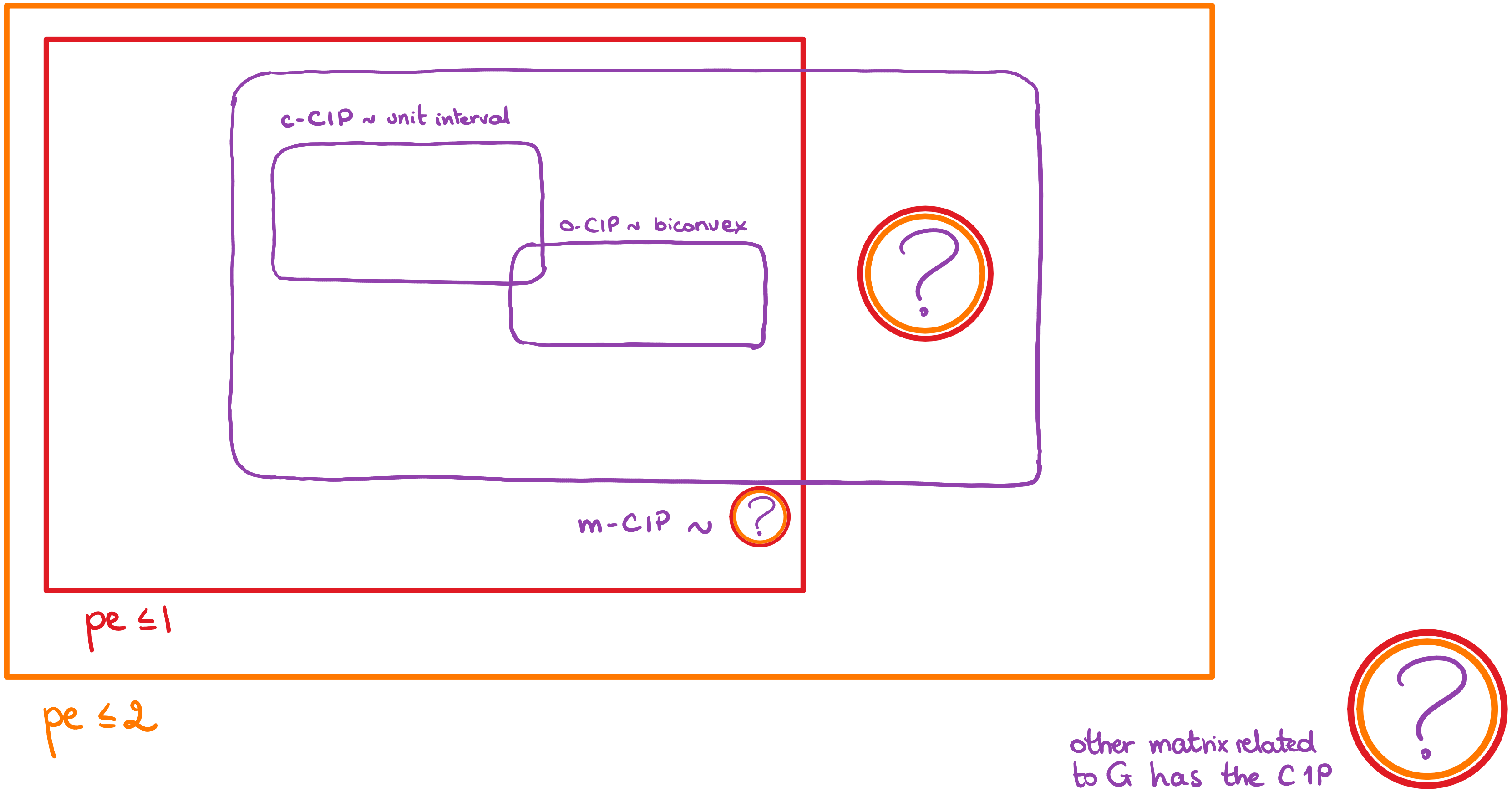
THE END?



THE END?


c-C1P ~ unit interval
o-C1P ~ biconvex
m-C1P ~ ?
pe ≤ 1
pe ≤ 2
other matrix related to G has the C1P
?

The end?



Thank you for your attention