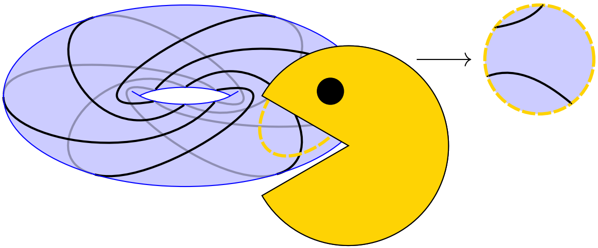
A Structural Approach to Tree Decompositions of Knots and Spatial Graphs

Corentin Lunel
Univ. Gustave Eiffel
Paris, France

Arnaud de Mesmay
Univ. Gustave Eiffel
CNRS, Paris, France



Knots

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A **knot** is a polygonal embedding $\mathbb{S}^1 \to \mathbb{S}^3$ considered up to ambient isotopy, i.e. continuous deformation without self intersection.

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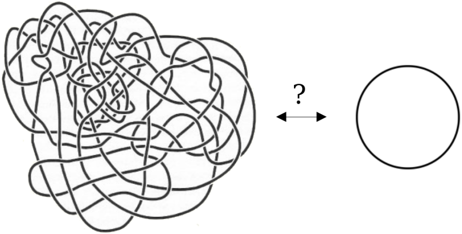
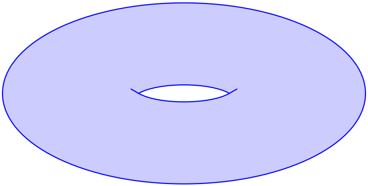


Natural algorithmic question :

Is a given knot equivalent to the trivial one ?
In **NP** ∩ **co-NP** [Hass-Lagarias-Pippenger 1999] and [Lackenby 2018].

Torus knots

A **torus knot** is a knot that can be embedded on the standard torus.



The standard torus.

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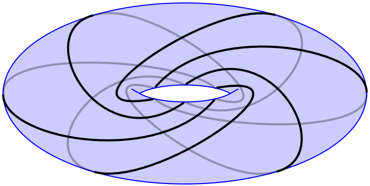

$T_{6,5}$, a torus knot.

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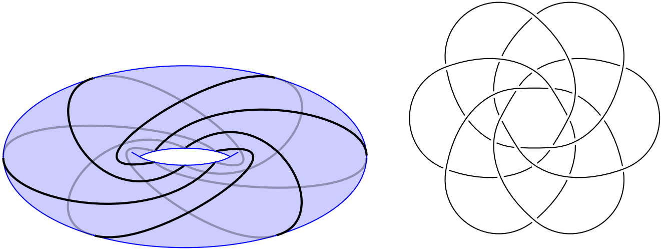


Two representations of $T_{6,5}$.

Knot diagram

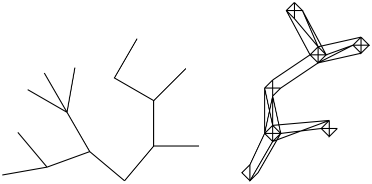
A **knot diagram** is a generic projection of a knot in the plane, it can be seen as a decorated 4-valent planar graph.

Treewidth of a graph

The **treewidth** aims to measure "how close" a graph is to a tree.

Some examples

- Small treewidth:



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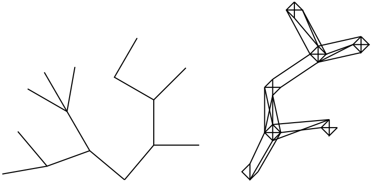
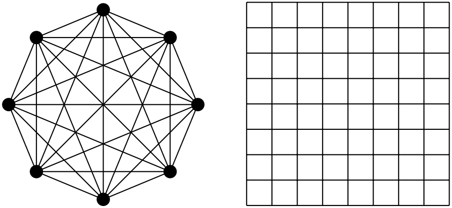


- High treewidth:



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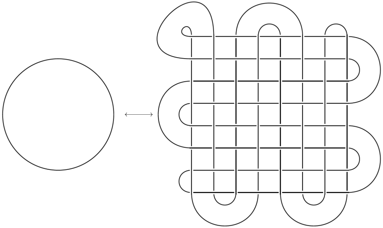

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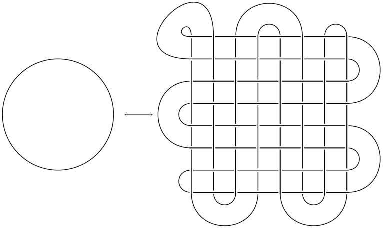

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Question from [Makowsky and Mariño, 2003] and [Burton, 2016]:

Are there knots for which all diagrams have high treewidth?

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Theorem [de Mesmay, Purcell, Schleimer, Sedgwick, 2019]:
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Our contribution

We introduce tools, inspired form structural graph theory, to answer that question positively:

- **Spherewidth**, a measure of how close a knot is to a tree, and a lower bound to treewidth.
- **Bubble tangles**, an optimal obstruction (lower bound) to spherewidth arising from embeddings on surfaces.

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- Key tool for **dynamic programming**.
  For example, on knots: Jones polynomial, Kauffman polynomial [Makowsky and Mariño, 2003], HOMFLY-PT polynomial [Burton, 2018], and quantum invariants [Maria, 2019] are efficiently computable on small treewidth diagrams of knots.

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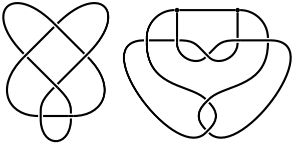
- Relevant in other areas of topology:
  For example: 3-manifolds [Huszár, Spreer, and Wagner, 2018], lower bounds distortion on knots [Pardon, 2011]...

Our contribution

Let K be a knot or **spatial graph** embedded in $\mathbb{S}^3$. Its **Spherewidth** is written $\mathrm{sw}(K)$.



Proposition $\mathrm{sw}(\mathrm{K}) \leq 2\mathrm{tw}(K)$.

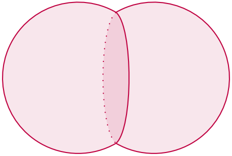
Theorem 1

Let n be the maximal order of a **bubble tangle** on K, $\mathrm{sw}(K) = n$.

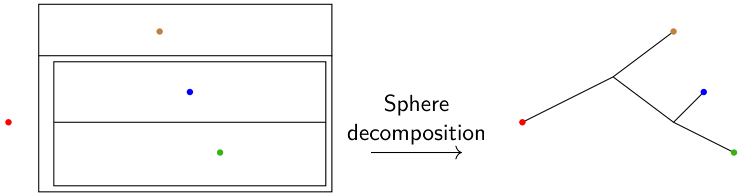
Theorem 2

Let K be a knot or spatial graph with the nested embeddings $K \hookrightarrow \Sigma \hookrightarrow \mathbb{S}^3$. Then there exists a **bubble tangle** of order $\frac{2}{3} \times crep(K, \Sigma)$.

Spherewidth definition



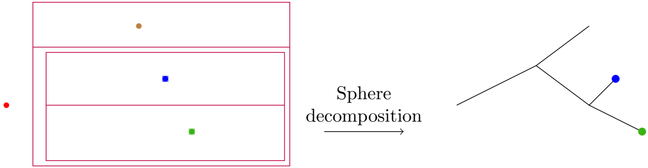
A **double bubble** : two spheres that intersect on a single disk.



Sphere decomposition

A **sphere decomposition** of $\mathbb{S}^3$ is a continuous map $f : \mathbb{S}^3 \longrightarrow T$ where T is a trivalent tree such that:

$$f^{-1} : \begin{cases} \text{leaf} & \mapsto \quad \text{point} \\ \text{vertex} & \mapsto \quad \text{double bubble} \\ \text{point interior to an edge} & \mapsto \quad \text{sphere} \end{cases}$$



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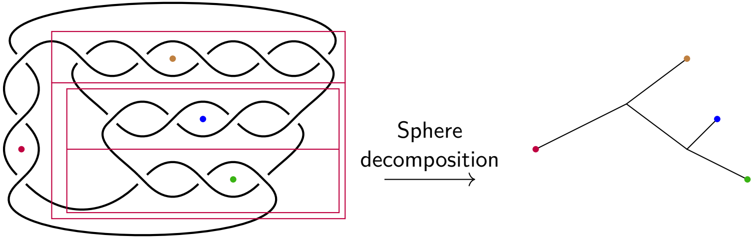
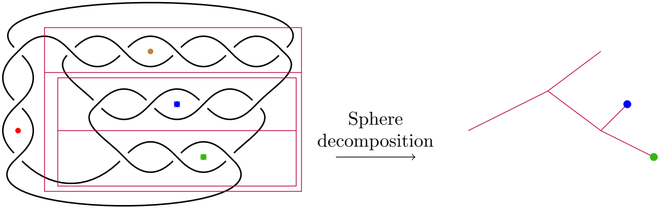


Spherewidth

The spherewidth of K written $\mathbf{sw}(K)$ is :

$$\text{sw}(K) = \inf_{f} \sup_{e \in E(T), x \in \mathring{e}} |f^{-1}(x) \cap K|.$$



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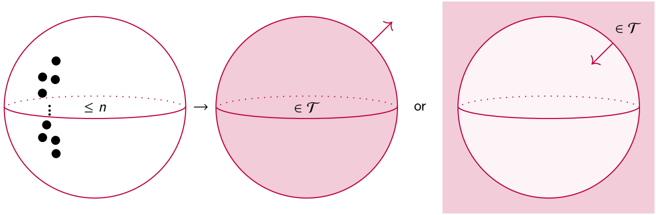
Proposition sw(K) $\leq 2\text{tw}(K)$.

Bubble tangle on knots

Bubble tangle

Let K be a knot, a **bubble tangle of order** n, denoted $\mathcal{T}$, is a collection of closed balls of $\mathbb{S}^3$ with less than n intersections with K on their boundaries such that:

- For any sphere S of $\mathbb{S}^3$, if $|S \cap K| < n$ then exactly one side of S is in $\mathcal{T}$.



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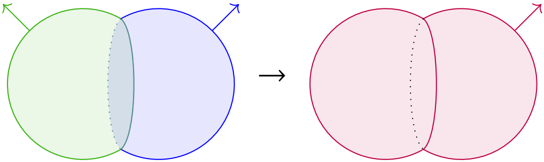
- For any three closed balls B_1, B_2, B_3 that induces a double bubble, not all three of B_1, B_2, B_3 are in $\mathcal{T}$.



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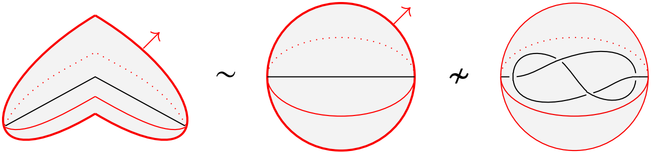
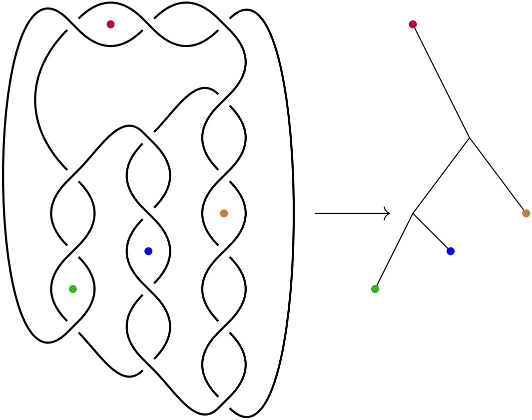
- For any three closed balls B_1, B_2, B_3 that induces a double bubble, not all three of B_1, B_2, B_3 are in $\mathcal{T}$.
- T contains trivial balls:



Results

Proposition

If there exists a **bubble tangle** of order n on K then $\mathrm{sw}(K) \geq n$.



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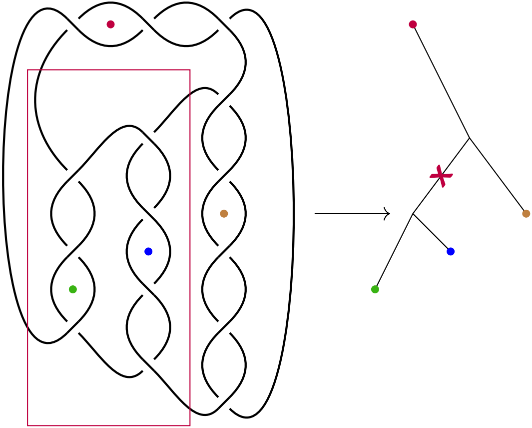

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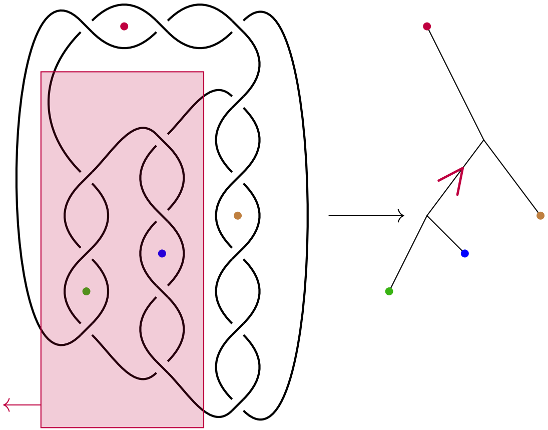

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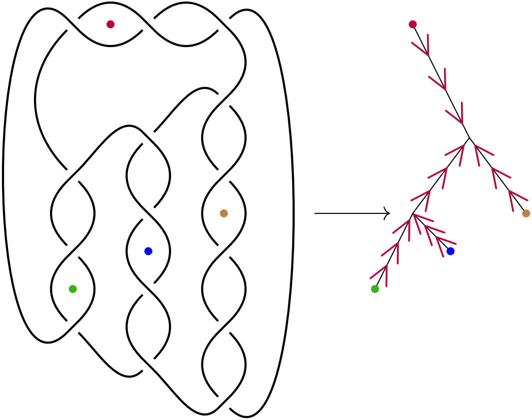

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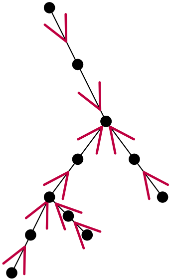

The orientation is consistent on half edges and necessarily has a sink.

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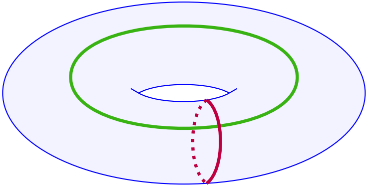
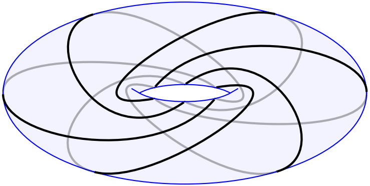
Bubble tangle from representativity

Compression representativity

If $K \hookrightarrow \Sigma \hookrightarrow \mathbb{S}^3$, the **compression representativity** $crep(K, \Sigma)$ of K on Σ is the minimum number of intersection between a non contractible, compressible curve on Σ and K.





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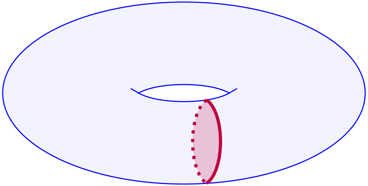

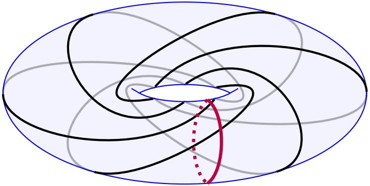

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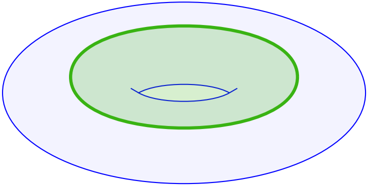
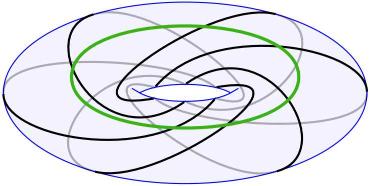
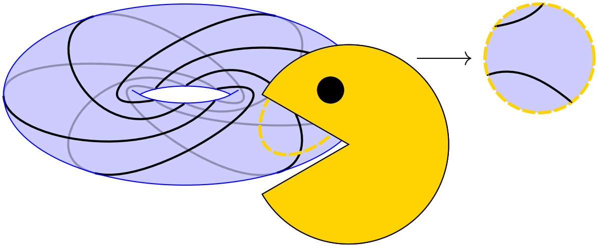


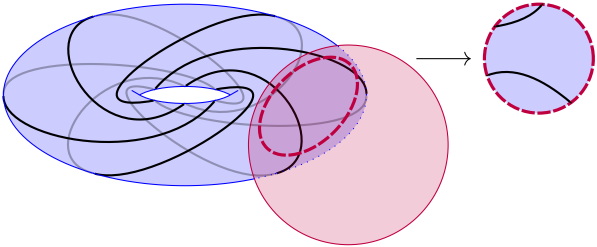


The compression representativity of the torus knot $T_{p,q}$ is $\min(p, q)$.

If the length of the mouth of pacman is less than the compression representativity, he can only eat a disc of the donut.



If a sphere intersects the knot less times than the compression representativity, then one of the sides of the sphere only contains discs of the surface.



Theorem 2

Let K be a knot or spatial graph embedded such that $K \hookrightarrow \Sigma \hookrightarrow \mathbb{S}^3$. Then there exists a **bubble tangle** of order $\frac{2}{3} \times crep(K, \Sigma)$.

Corollary Torus knots have high treewidth.

Perspectives

Questions:

- Can we do better than $\frac{2}{3}$ in theorem 2?
  We conjecture $\frac{4}{3}$ for torus knots.
- Are there more concepts to import from structural graph theory?
- Can we design FPT algorithms on knots using spherewidth?
- Can we compute some graph invariant on spatial graphs using spherewidth?
  For example carving width on linkless graphs.

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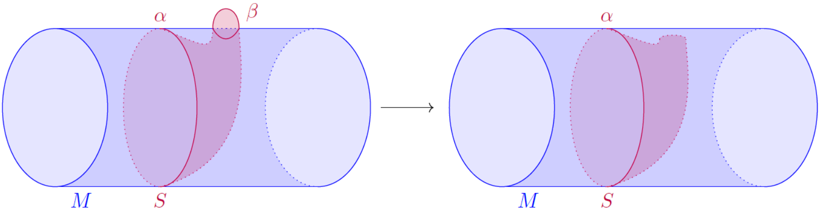
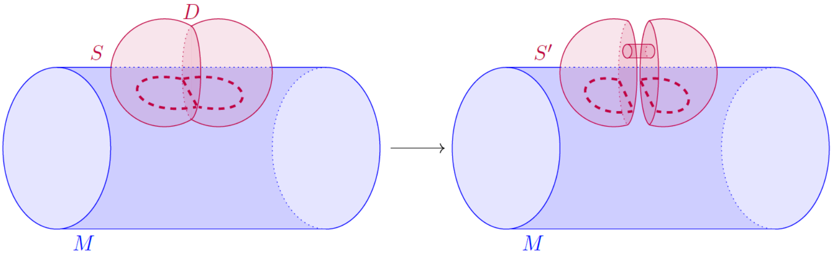
Thank you for listening! Questions?



Removing an inessential curve from S.



Transformation from a double bubble to S'