A Structural Approach to Tree Decompositions of Knots and Spatial Graphs

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Knots

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Natural algorithmic question:
Is a given knot equivalent to the trivial one?
In $\text{NP} \cap \text{co-NP}$ [Hass-Lagarias-Pippenger 1999] and [Lackenby 2018].
**Torus knots**

A **torus knot** is a knot that can be embedded on the standard torus.

The standard torus.
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\[ T_{6,5}, \text{ a torus knot.} \]
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Two representations of $T_{6,5}$.

Knot diagram

A knot diagram is a generic projection of a knot in the plane, it can be seen as a decorated 4-valent planar graph.
Treewidth of a graph

The **treewidth** aims to measure "how close" a graph is to a tree.

Some examples

- Small treewidth:
Treewidth of a graph

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- Small treewidth:

- High treewidth:
A question

Treewidth of knots

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Question from [Makowsky and Mariño, 2003] and [Burton, 2016]:

Are there knots for which all diagrams have high treewidth?
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Theorem [de Mesmay, Purcell, Schleimer, Sedgwick, 2019]:
Let $T_{p,q}$ be a torus knot. Then $tw(T_{p,q}) = \Omega(\min(p, q))$. 
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**Theorem** [de Mesmay, Purcell, Schleimer, Sedgwick, 2019]:
Let $T_{p,q}$ be a torus knot. Then $\text{tw}(T_{p,q}) = \Omega(\min(p, q))$.

**Our contribution**
We introduce tools, inspired form structural graph theory, to answer that question positively:

- **Spherewidth**, a measure of how close a knot is to a tree, and a lower bound to treewidth.
- **Bubble tangles**, an optimal obstruction (lower bound) to spherewidth arising from embeddings on surfaces.
Why is treewidth interesting?

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- Key tool for **dynamic programming**. For example, on knots: Jones polynomial, Kauffman polynomial [Makowsky and Mariño, 2003], HOMFLY-PT polynomial [Burton, 2018], and quantum invariants [Maria, 2019] are efficiently computable on small treewidth diagrams of knots.
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- Relevant in other areas of topology:
  For example: 3-manifolds [Huszár, Spreer, and Wagner, 2018], lower bounds distortion on knots [Pardon, 2011]...
Our contribution

Let $K$ be a knot or spatial graph embedded in $S^3$. Its Spherewidth is written $\text{sw}(K)$.

\begin{center}
\includegraphics[width=0.5\textwidth]{knot-diagram.png}
\end{center}

Proposition $\text{sw}(K) \leq 2\text{tw}(K)$.

**Theorem 1**

Let $n$ be the maximal order of a bubble tangle on $K$, $\text{sw}(K) = n$.

**Theorem 2**

Let $K$ be a knot or spatial graph with the nested embeddings $K \hookrightarrow \Sigma \hookrightarrow S^3$. Then there exists a bubble tangle of order $\frac{2}{3} \times \text{crep}(K, \Sigma)$. 

Spherewidth definition

A **double bubble**: two spheres that intersect on a single disk.
Sphere decomposition

A sphere decomposition of $S^3$ is a continuous map $f : S^3 \rightarrow T$ where $T$ is a trivalent tree such that:

$$f^{-1} : \begin{cases} 
\text{leaf} & \mapsto \text{point} \\
\text{vertex} & \mapsto \text{double bubble} \\
\text{point interior to an edge} & \mapsto \text{sphere}
\end{cases}$$
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The spherewidth of $K$ written $sw(K)$ is:

$$sw(K) = \inf_{f} \sup_{e \in E(T), x \in \partial e} |f^{-1}(x) \cap K|.$$
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**Proposition** $\text{sw}(K) \leq 2\text{tw}(K)$. 
Bubble tangle on knots

Bubble tangle

Let $K$ be a knot, a **bubble tangle of order** $n$, denoted $\mathcal{T}$, is a collection of closed balls of $S^3$ with less than $n$ intersections with $K$ on their boundaries such that:

- For any sphere $S$ of $S^3$, if $|S \cap K| < n$ then exactly one side of $S$ is in $\mathcal{T}$.
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- For any three closed balls $B_1, B_2, B_3$ that induces a double bubble, not all three of $B_1, B_2, B_3$ are in $\mathcal{T}$.
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- For any three closed balls $B_1, B_2, B_3$ that induces a double bubble, not all three of $B_1, B_2, B_3$ are in $\mathcal{T}$.
- $\mathcal{T}$ contains trivial balls:
Results

Proposition
If there exists a bubble tangle of order $n$ on $K$ then $sw(K) \geq n$. 
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If there exists a **bubble tangle** of order $n$ on $K$ then $\text{sw}(K) \geq n$.

The orientation is consistent on half edges and necessarily has a sink.
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Theorem 1
Let $n$ be the maximal order of a bubble tangle on $K$, $sw(K) = n$. 
Bubble tangle from representativity

Compression representativity

If $K \leftrightarrow \Sigma \leftrightarrow \mathbb{S}^3$, the compression representativity $crep(K, \Sigma)$ of $K$ on $\Sigma$ is the minimum number of intersection between a non contractible, compressible curve on $\Sigma$ and $K$. 
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The compression representativity of the torus knot $T_{p,q}$ is $\min(p, q)$. 
If the length of the mouth of pacman is less than the compression representativity, he can only eat a disc of the donut.
If a sphere intersects the knot less times than the compression representativity, then one of the sides of the sphere only contains discs of the surface.

**Theorem 2**
Let \( K \) be a knot or spatial graph embedded such that \( K \hookrightarrow \Sigma \hookrightarrow S^3 \). Then there exists a **bubble tangle** of order \( \frac{2}{3} \times \text{crep}(K, \Sigma) \).

**Corollary** Torus knots have high treewidth.
Perspectives

Questions:

• Can we do better than $\frac{2}{3}$ in theorem 2? We conjecture $\frac{4}{3}$ for torus knots.

• Are there more concepts to import from structural graph theory?

• Can we design FPT algorithms on knots using spherewidth?

• Can we compute some graph invariant on spatial graphs using spherewidth? For example carving width on linkless graphs.
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Thank you for listening! Questions?
Removing an inessential curve from $S$. 
Transformation from a double bubble to $S'$