TOUGH GRAPHS AND HAMILTONIAN DEGREE CONDITIONS

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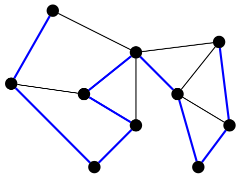
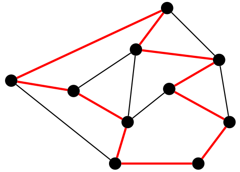
Some definitions

Definition

Hamiltonian cycle (resp. path): cycle (resp. path) containing all vertices of the graph exactly once.





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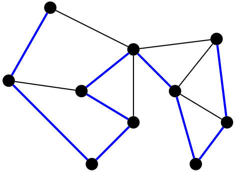

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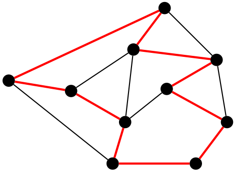

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A graph is *Hamiltonian* if it contains a Hamiltonian cycle.

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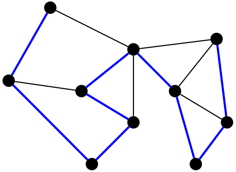

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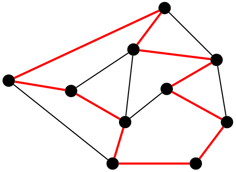

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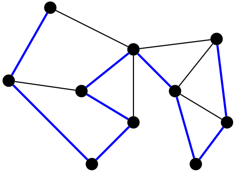

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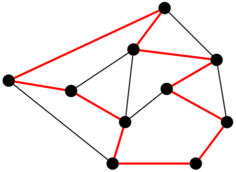


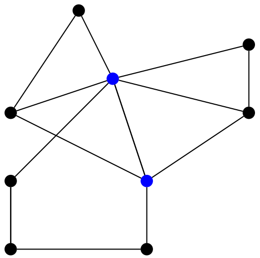
Determining if a graph is Hamiltonian is NP-complete.

Property

Every graph with a vertex cut-set is non-Hamiltonian

Cut-sets

What if S is a cut-set such that $|S| \geq 2$?



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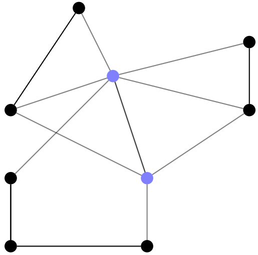

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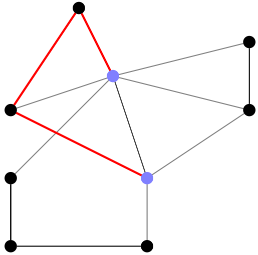

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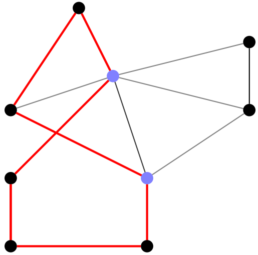

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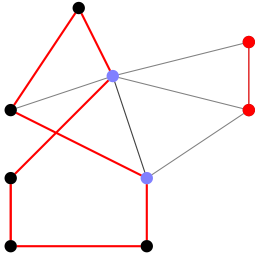

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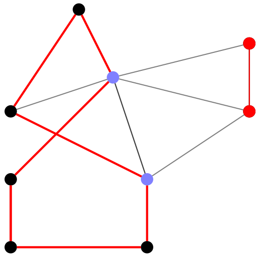

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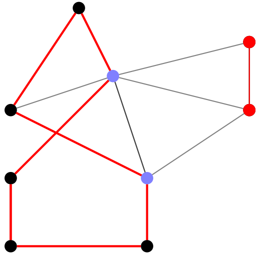
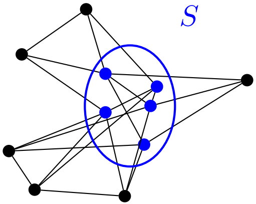


$c(H)$ is the number of connected components of a graph H.
Is there a link between $c(G[V \setminus S])$ and $|S|$ for a Hamiltonian graph?

t-tough graph

Definition

A graph G is t-tough if, for all subsets of vertices S, $t \times c(G[V \setminus S]) \leq |S|$



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$$c(G[V \setminus S]) = 3$$
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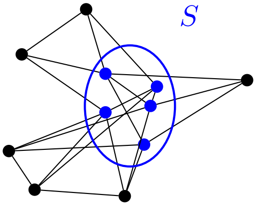

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$c(G[V \setminus S]) = 3$
G is not 2-tough

Conjecture (Chvàtal, 1973)

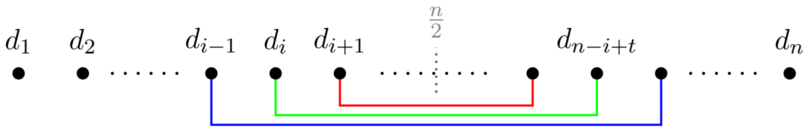
$\exists t$ such that every t-tough graph is Hamiltonian.

The property $P(t)$

Degree sequence: $d_1 \leq d_2 \leq \cdots \leq d_n$ such that d_i is the degree of v_i $\forall v_i \in V(G)$.

Definition

G is $P(t)$ if $\forall i < \frac{n}{2}$ $d_i \leq i \Rightarrow d_{n-i+t} \geq n - i$.



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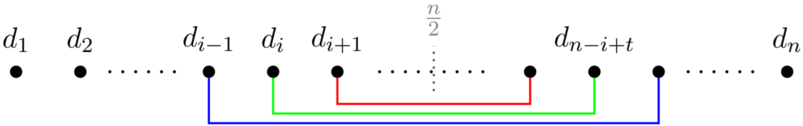

Theorem (Chvàtal 1972)

If $\forall i < \frac{n}{2}$ $d_i \leq i \Rightarrow d_{n-i} \geq n - i$, then G is Hamiltonian.

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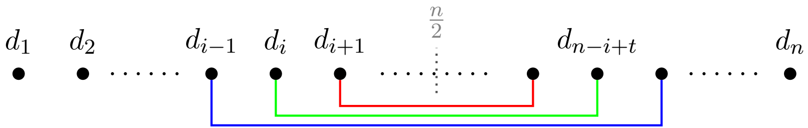

Theorem (Chvàtal 1972)

If G is $P(0)$ then G is Hamiltonian.

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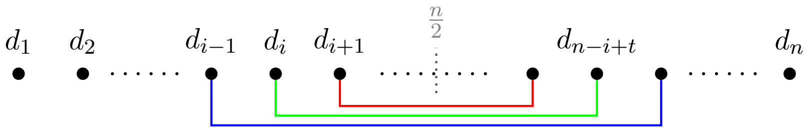

Conjecture (Hoàng 1995)

If G is t-tough and $P(t)$ then G is Hamiltonian.

The property $P(t)$

Degree sequence: $d_1 \leq d_2 \leq \cdots \leq d_n$ such that d_i is the degree of v_i $\forall v_i \in V(G)$.

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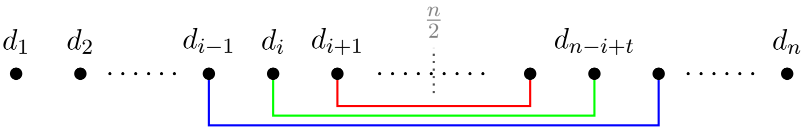

Theorem (Hoàng 1995)

For $t \leq 3$, if G is t-tough and $P(t)$ then G is Hamiltonian.

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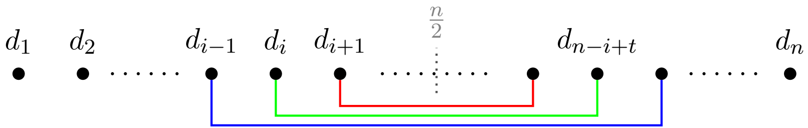

Theorem (Hoàng and *Robin* 2023)

For $t \leq 4$, if G is t-tough and $P(t)$ then G is Hamiltonian.

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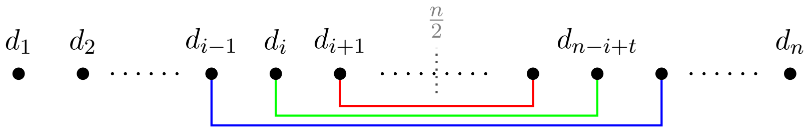


Theorem (Hoàng and *Robin* 2023)

For $t \leq 4$, if G is t-tough and $P(t)$ then G is Hamiltonian.

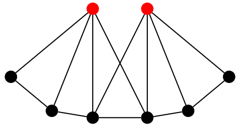
$\Rightarrow$ We need to extend the closure lemma: the t-closure lemma

The Closure Lemma

Definition

G^* is the *closure* of G if :

1. $V(G) = V(G^*)$
2. $E(G^*) = E(G) \cup \{xy : d(x) + d(y) \geq n\}$



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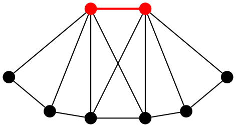

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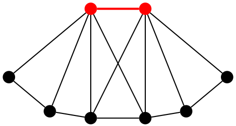

The Closure Lemma (Bondy and Chvatàl, 1976)

A graph G is Hamiltonian if and only if its closure G^* is Hamiltonian.

The t-closure Lemma

Definition

G^* is the *t-closure* of G if :

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The t-closure Lemma (Hoàng and Robin, 2023)

A t-tough graph G is Hamiltonian if and only if its $\frac{2t+1}{3}$closure G^* is Hamiltonian. ($t \geq 2$).

Proof of the 1-closure lemma

The 1-closure lemma (Hoàng and Robin, 2023)

If G is 2-tough then G is Hamiltonian if and only if its 1-closure is.

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The 1-closure lemma (Hoàng and Robin, 2023)

G is 2-tough and $x, y \in V$ such that $xy \notin E$ and $d(x) + d(y) \geq n - 1$
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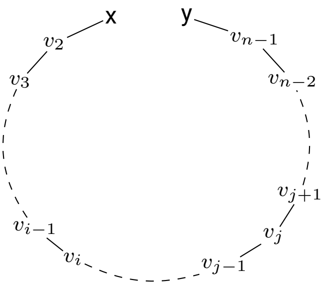

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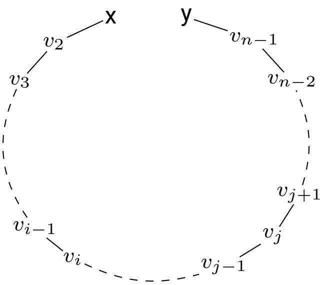

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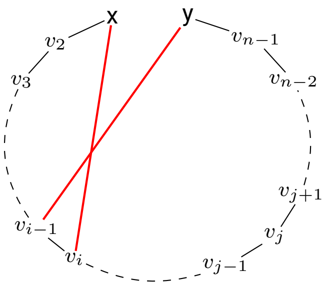

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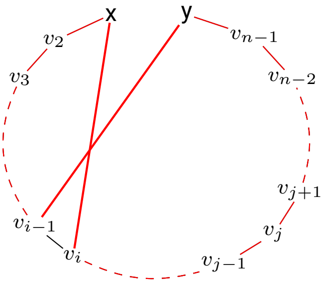

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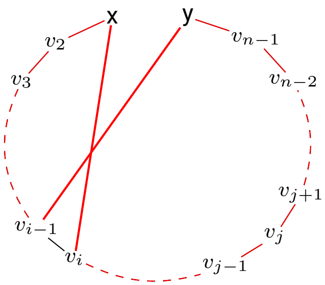

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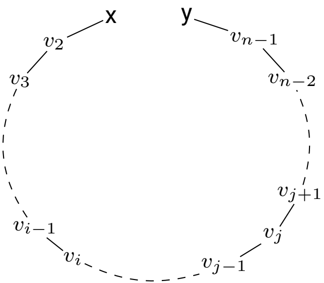

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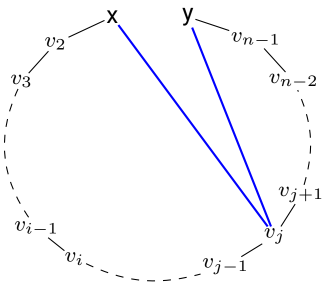

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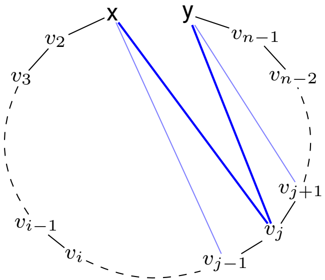

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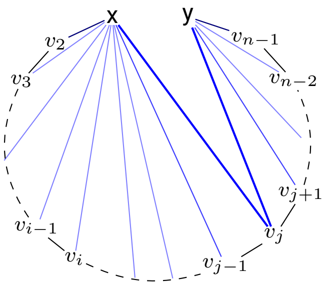

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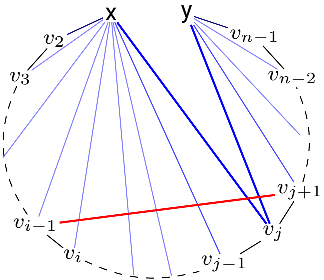

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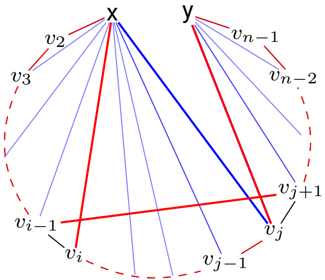

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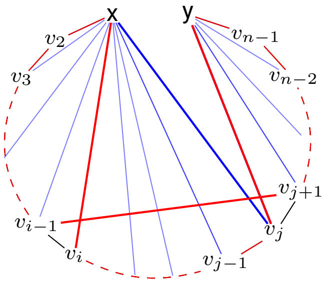

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Proof of the 1-closure lemma

GREYC Electronics and Computer Science Laboratory

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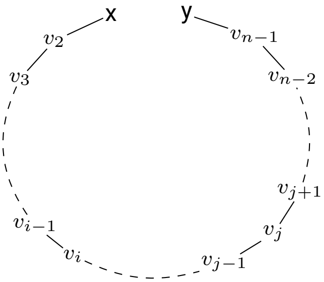

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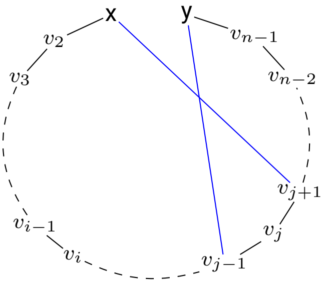

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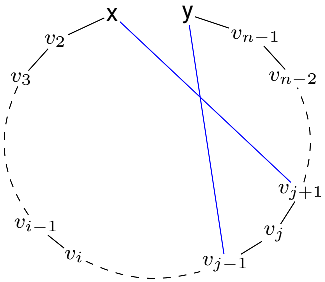

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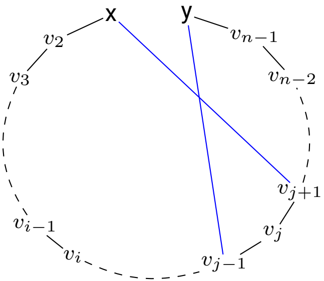

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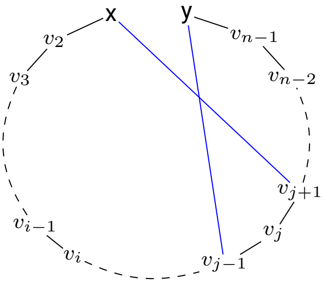

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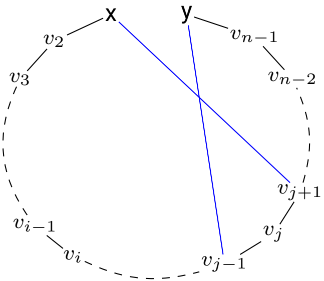

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- G is 2-tough $\Leftrightarrow 2c(G[V \setminus N(S)]) \leq |N(S)|$, a contradiction.

4-tough and $P(4)$

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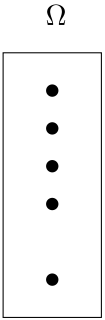

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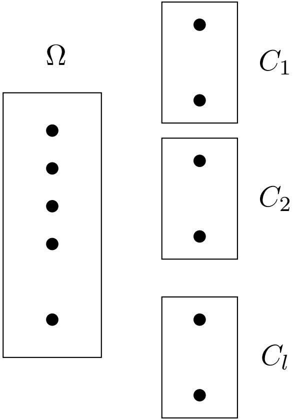

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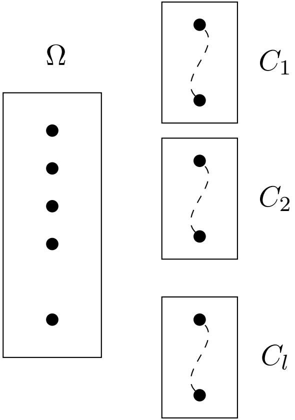

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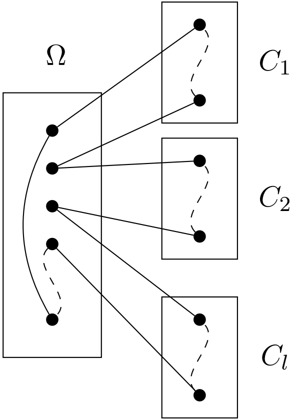

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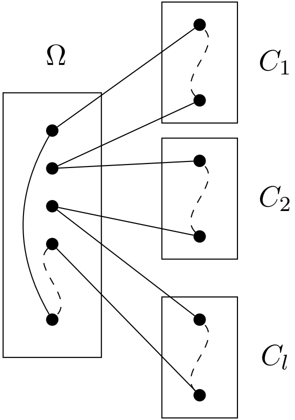

And $t \geq 5$?

Conclusion

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For $t \leq 4$, if G is t-tough and $P(t)$ then G is Hamiltonian

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A t-tough graph G is Hamiltonian if and only if its $\frac{2t+1}{3}$-closure G^* isHamiltonian. $(t \geq 2)$.

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Thank you !