The Grid-Minor Theorem revisited

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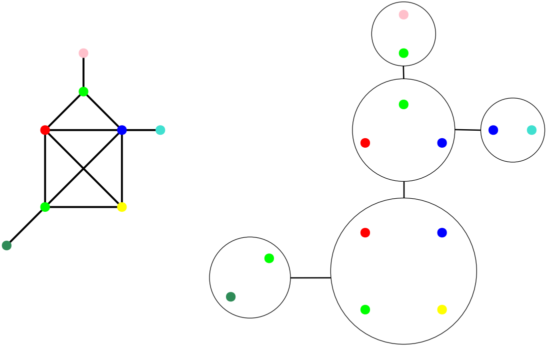
Treewidth

Tree-decomposition: $(T, (B_x \mid x \in V(T)))$ such that

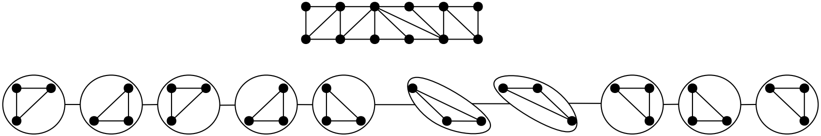
1. T is a tree,
2. $\bigcup_x B_x = V(G)$,
3. $\forall uv \in E(G), \exists x \in V(T)$ s.t. $u, v \in B_x$,
4. $\{x \in V(T) \mid u \in B_x\}$ is a connected subtree, $\forall u \in V(G)$.



width $= \max |B_x| - 1$.
Treewidth: $\mathrm{tw}(G) =$ min width of a tree-decomposition of G.

Pathwidth and treedepth

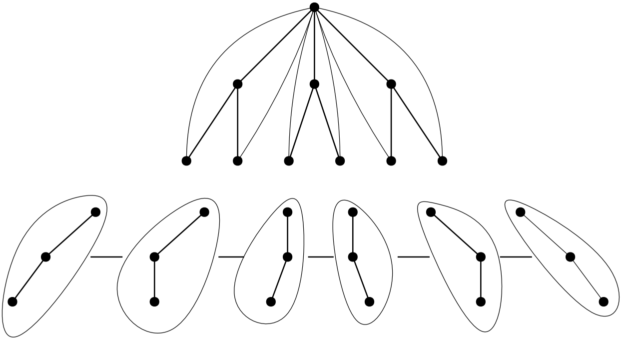
Pathwidth: $\mathrm{pw}(G)$ = minimum width of a *path*-decomposition.



Pathwidth and treedepth

Pathwidth: $\mathrm{pw}(G)$ = minimum width of a *path*-decomposition.

Treedepth: $\mathrm{td}(G)$ = minimum depth of a forest whose completion contains G.



$$\mathrm{tw}(G) \leqslant \mathrm{pw}(G) \leqslant \mathrm{td}(G) - 1$$

Minor

Definition

H is a **minor** of G if H can be obtained from G by successive

1. vertex deletions
2. edge deletions
3. edge contractions.

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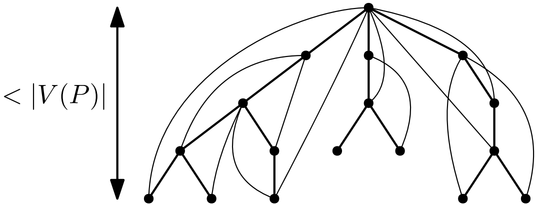
Question: for which X, X-minor-free graphs have bounded td? pw? tw?

Two fundamental theorems

Proposition (Nešetřil and Ossona de Mendez, 2005)

For every **path** P, $\exists f(P)$ such that every P-minor-free graph G has $\mathrm{td}(G) \leqslant f(P)$.

Proof: consider a DFS tree



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Grid-Minor Theorem (Robertson and Seymour, 1986)

For every **planar** graph X, $\exists f(X)$ such that every X-minor-free graph G has $\mathrm{tw}(G) \leqslant f(X)$.

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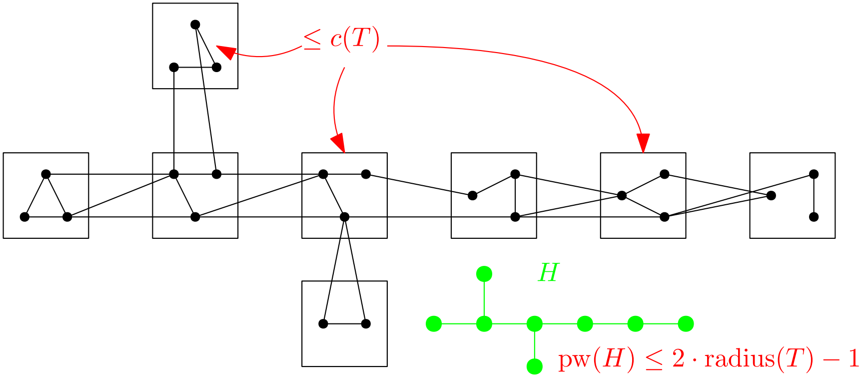
Remark: $K_{|V(X)|-1}$ is X-minor-free so $f(X) \geqslant |V(X)| - 2$.
$\hookrightarrow$ $\max\{\mathrm{tw}(G) \mid G$ X-minor-free$\}$ is tied to $|V(X)|$.

Product structure strengthening

Theorem (Dujmović, Hickingbotham, Joret, Micek, Morin, Wood, 2023)

For every **tree** T, $\exists c(T)$ such that for every T-minor-free graph G, $G \subseteq H \boxtimes K_{c(T)}$ where $\mathrm{pw}(H) \leqslant 2\,\mathrm{radius}(T) - 1$.

Moreover, $\mathrm{radius}(T)$ is the right parameter.

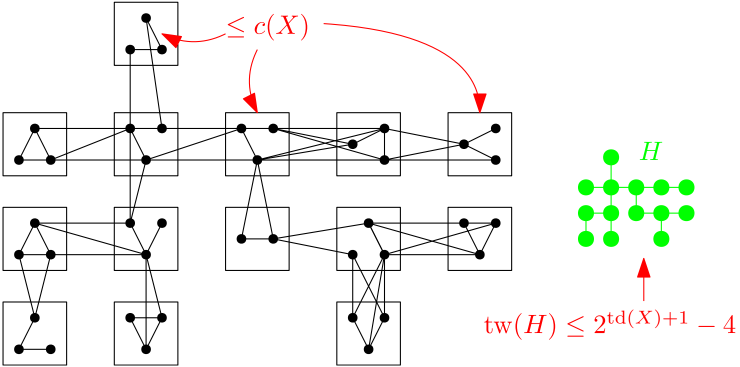


Product structure strengthening

Theorem (DHHJLMMRW, 2023)

For every **planar** X, $\exists c(X)$ such that for every X-minor-free graph G, $G \subseteq H \boxtimes K_{c(X)}$ where $\mathrm{tw}(H) \leqslant 2^{\mathrm{td}(X)+1} - 4$.

Moreover, $\mathrm{td}(X)$ is the right parameter.



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Remark: this implies RS's Grid-Minor Theorem since

$$\begin{aligned}\mathrm{tw}(G) &\leqslant \mathrm{tw}(H \boxtimes K_{c(X)}) \\ &\leqslant (\mathrm{tw}(H) + 1)c(X) - 1 \\ &\leqslant 2^{\mathrm{td}(X)+1}\, c(X) =: f(X).\end{aligned}$$

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Disclaimer: we use the Grid-Minor Theorem in the proof.

Product structure for graphs of bounded treewidth

Theorem (DHHJLMMRW, 2023)

For every X, for every X-minor-free G with $\mathrm{tw}(G) < t$, $G \subseteq H \boxtimes K_{c(X)\cdot t}$ where $\mathrm{tw}(H) \leqslant 2^{\mathrm{td}(X)+1} - 4$.

Together with the Grid-Minor Theorem (RS, 86), this implies the main result.

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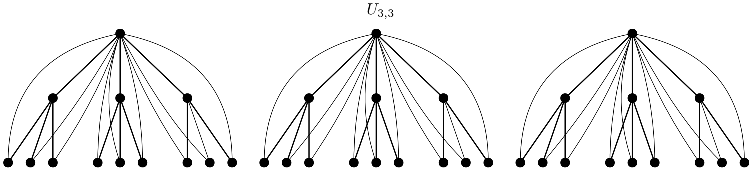
$U_{h,d}$ = completion of the complete d-ary forest of depth h



Theorem (DHHJLMMRW, 2023)

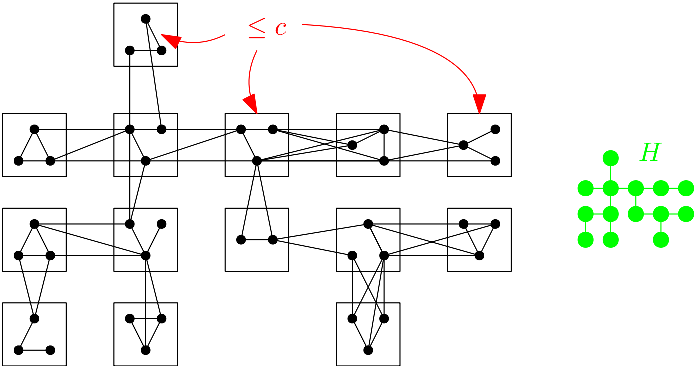
For every $U_{h,d}$-minor-free G with $\mathrm{tw}(G) < t$, $G \subseteq H \boxtimes K_{c(h,d)\cdot t}$ where $\mathrm{tw}(H) \leqslant 2^{h+1} - 4$.

Factorization and partition

Observation

$G \subseteq H \boxtimes K_c$ if and only if there is a partition $\mathcal{P}$ of $V(G)$ such that

- $|P| \leqslant c$ for all $P \in \mathcal{P}$,
- $G/\mathcal{P} \subseteq H$.



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Reformulation:

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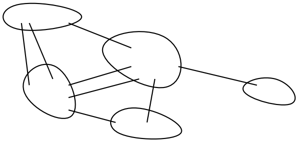
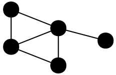
Important tool: attached models

Models:



model of



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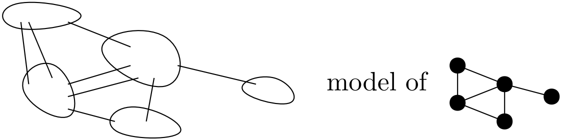

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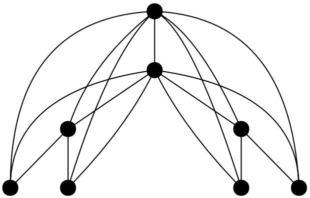
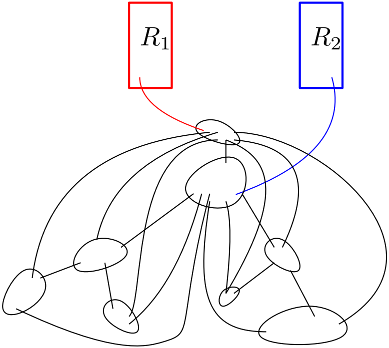
$(R_1, \ldots, R_k)$-**attached model** of $K_k \oplus G$

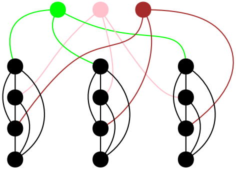
$K_2 \oplus U_{2,2}$





Proof sketch: building $U_{h,d}$ minors

three attached $K_4 \oplus U_{0,3}$



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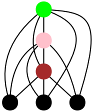

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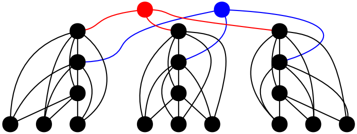

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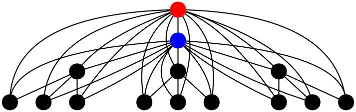

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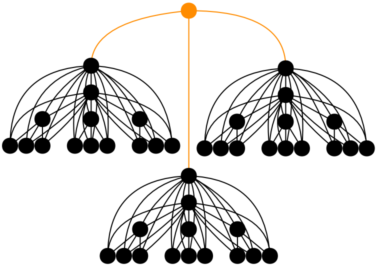
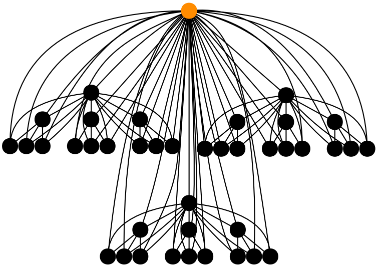


$K_1 \oplus U_{3,3}$



Proof sketch: main induction

We prove by induction on h the following

Property

For every $K_k \oplus U_{h,d}$-minor-free graph G with $\mathrm{tw}(G) < t$, there is a partition $\mathcal{P}$ such that

- $|P| \leqslant c(h, d, k) \cdot t, \forall P \in \mathcal{P}$,
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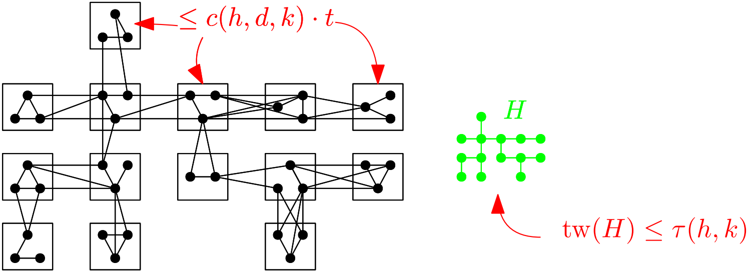

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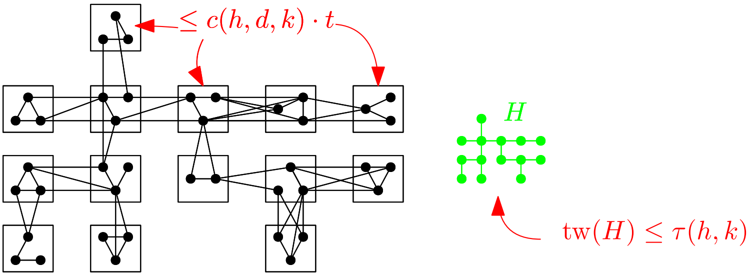
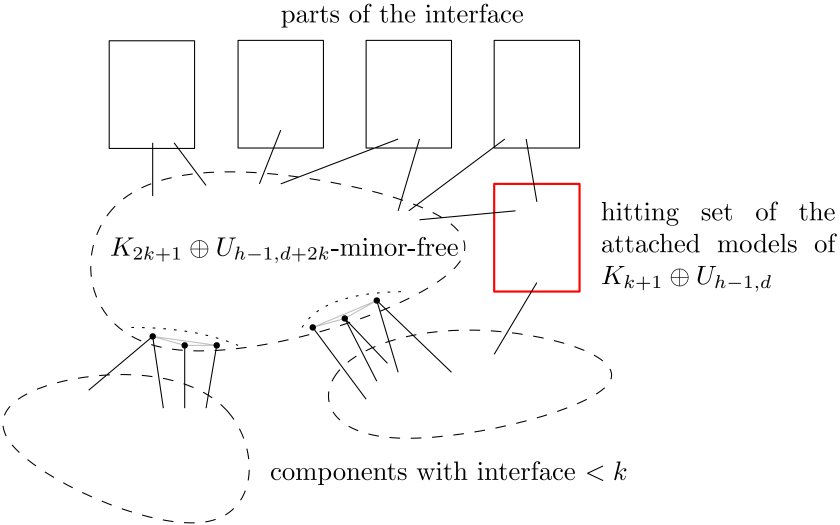


Base case: about K_k-minor-free graphs. Proved by Illingworth, Scott, Wood, 2022.

Proof sketch: main induction

G is $K_k \oplus U_{h,d}$-minor-free



Some other results using the same method

Excluding an apex:

Theorem (DHHJLMMRW, 2023)

For every **apex** graph X, for every X-minor-free graph G,
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Weak coloring numbers:

Theorem (DHHJLMMRW, 2023)

For every X, for every X-minor-free graph G,

$$\mathrm{wcol}_r(G) = \mathcal{O}_X\left(r^{2^{\mathrm{td}(X)+1}-3}\right).$$

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Thank you!